MICHAEL STRUWE

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Variational Methods

Applications
to Nonlinear Partial Differential Equations
and Hamiltonian Systems

Fourth Edition

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Michael Struwe

Variational Methods

Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems

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Preface to the Fourth Edition

Almost twenty years after conception of the first edition, it was a challenge to prepare an updated version of this text on the Calculus of Variations. The field has truely advanced dramatically since that time, to an extent that I find it impossible to give a comprehensive account of all the many important developments that have occurred since the last edition appeared. Fortunately, an excellent overview of the most significant results, with a focus on functional analytic and Morse theoretical aspects of the Calculus of Variations, can be found in the recent survey paper by Ekeland-Ghoussoub [1]. I therefore have only added new material directly related to the themes originally covered.

Even with this restriction, a selection had to be made. In view of the fact that flow methods are emerging as the natural tool for studying variational problems in the field of Geometric Analysis, an emphasis was placed on advances in this domain. In particular, the present edition includes the proof for the convergence of the Yamabe flow on an arbitrary closed manifold of dimension $3 \leq m \leq 5$ for initial data allowing at most single-point blow-up. Moreover, we give a detailed treatment of the phenomenon of blow-up and discuss the newly discovered results for backward bubbling in the heat flow for harmonic maps of surfaces.

Aside from these more significant additions, a number of smaller changes have been made throughout the text, thereby taking care not to spoil the freshness of the original presentation. References have been updated, whenever possible, and several mistakes that had survived the past revisions have now been eliminated. I would like to thank Silvia Cingolani, Irene Fonseca, Emmanuel Hebey, and Maximilian Schultz for helpful comments in this regard. Moreover, I am indebted to Gilles Angelsberg, Ruben Jakob, Reto Müller, and Melanie Rupflin, for carefully proof-reading the new material.

Zürich, July 2007 *Michael Struwe*

Preface to the Third Edition

The Calculus of Variations continues to be an area of very rapid growth. Variational methods are indispensable as a tool in mathematical physics and geometry.

Results on Ginzburg-Landau type variational problems inspire research on the related Seiberg-Witten functional on a Kähler surface and invite speculations about possible applications in topology (Ding-Jost-Li-Peng-Wang [1]).

Variational methods are applied in cosmology, as in the recent work of Fortunato-Giannoni-Masiello [1] and Giannoni-Masiello-Piccione [1] on geodesics in Lorentz manifolds and gravitational lenses.

Applications to Hamiltonian dynamics now include a proof of the Seifert conjecture on brake orbits (Giannoni [1]) and results on homoclinic and heteroclinic solutions (Coti Zelati-Ekeland-Séré [1], Rabinowitz [1], Séré [1]) with interesting counterparts in the field of semilinear elliptic equations (Coti-Zelati-Rabinowitz [1], Rabinowitz [13]).

The Calculus of Variations also has advanced on a more technical level. Campa-Degiovanni [1], Corvellec-Degiovanni-Marzocchi [1], Degiovanni-Marzocchi [1], Ioffe [1], and Ioffe-Schwartzman [1] have extended critical point theory to functionals on metric spaces, with applications, for instance, to quasilinear elliptic equations (Arioli [1], Arioli-Gazzola [1], Canino-Degiovanni [1]).

Bolle [1] has proposed a new approach to perturbation theory, as treated in Section II.7 of this monograph. Numerous applications are studied in Bolle-Ghoussoub-Tehrani [1].

The method of parameter dependence as in Sections I.7 and II.9 has found further striking applications in Chern-Simons theory (Struwe-Tarantello [1]) and independently for a related problem in mean field theory (Ding-Jost-Li-Wang [1]). Inspired by these results, Wang-Wei [1] were able to solve a problem in chemotaxis with a similar structure. Jeanjean [1] and Jeanjean-Toland [1] have discovered an abstract setting where parameter dependence may be exploited.

Ambrosetti [1], Ambrosetti-Badiale-Cingolani [1], and Ambrosetti-Badiale [1], [2] have found new applications of variational methods in bifurcation theory, refining the classical results of Böhme [1] and Marino [1]. In Ambrosetti-Garcia Azorero-Peral [1] these ideas are applied to obtain precise existence results for conformal metrics of prescribed scalar curvature close to a constant, which shed new light on the work of Bahri-Coron [1], [2], Chang-Yang [1] quoted in Section III.4.11.

The field of critical equations as in Chapter III has been particularly active.

Concentration profiles for Palais-Smale sequences as in Theorem III.3.1 have been studied in more detail by Rey [1] and Flucher [1].

Quite surprisingly, results analogous to Theorem III.3.1 have been discovered also for sequences of solutions to critical semilinear wave equations (Bahouri-Gérard [1]).

For the semilinear elliptic equations of critical exponential growth related to the Moser-Trudinger inequality on a planar domain the patterns for existence and non-existence results are strikingly analogous to the higher dimensional case (Adimurthi [1], Adimurthi-Srikanth-Yadava [1]), and, on a macroscopic scale, quantization phenomena analogous to Theorem III.3.1 are observed for concentrating solutions of semilinear equations with exponential growth (Brezis-Merle [1], Li-Shafrir [1]). However, results of Struwe [17] and Ogawa-Suzuki [1] on the one hand and an example by Adimurthi-Prashanth [1] on the other suggest that there may be many qualitatively distinct types of blow-up behavior for Palais-Smale sequences in this case. Still, Theorem III.3.1 remains valid for solutions (Adimurthi-Struwe [1]) and also the analogue of Theorem III.3.4 has been obtained (Struwe [25]). The many similarities and subtle differences to the critical semilinear equations in higher dimensions make this field particularly attractive for further study.

References have been updated and a small number of mistakes have been rectified. I am indepted to Gerd Müller, Paul Rabinowitz, and Henry Wente for their comments.

Zürich, July 1999 *Michael Struwe*

Preface to the Second Edition

During the short period of five years that have elapsed since the publication of the first edition a number of interesting mathematical developments have taken place and important results have been obtained that relate to the theme of this book.

First of all, as predicted in the Preface to the first edition, Morse theory, indeed, has gone through a dramatic change, influenced by the work by Andreas Floer on Hamiltonian systems and in particular, on the Arnold conjecture. There are now also excellent accounts of these developments and their ramifications; see, in particular, the monograph by Matthias Schwarz [1]. The book by Hofer-Zehnder [2] on Symplectic Geometry shows that variational methods and, in particular, Floer theory have applications that range far beyond the classical area of analysis.

Second, as a consequence of an observation by Stefan Müller [1] which prompted the seminal work of Coifman-Lions-Meyer-Semmes [1], Hardy spaces and the space BMO are now playing a very important role in weak convergence results, in particular, when dealing with problems that exhibit a special (determinant) structure. A brief discussion of these results and some model applications can be found in Section I.3.

Moreover, variational problems depending on some real parameter in certain cases have been shown to admit rather surprising a-priori bounds on critical points, with numerous applications. Some examples will be given in Sections I.7 and II.9.

Other developments include the discovery of Hamiltonian systems with no periodic orbits on some given energy hypersurface, due to Ginzburg and Herman, and the discovery, by Chang-Ding-Ye, of finite time blow-up for the evolution problem for harmonic maps of surfaces, thus completing the results in Sections II.8, II.9 and III.6, respectively.

A beautiful recent result of Ye concerns a new proof of the Yamabe theorem in the case of a locally conformally flat manifold. This proof is presented in detail in Section III.4 of this new edition.

In view of their numerous and wide-ranging applications, interest in variational methods is very strong and growing. Out of the large number of recent publications in the general field of the calculus of variations and its applications some 50 new references have been added that directly relate to one of the themes in this monograph.

Owing to the very favorable response with which the first edition of this book was received by the mathematical community, the publisher has suggested that a second edition be published in the Ergebnisse series. It is a pleasure to thank all the many mathematicians, colleagues, and friends who

have commented on the first edition. Their enthusiasm has been highly inspiring. Moreover, I would like to thank, in particular, Matts Essen, Martin Flucher and Helmut Hofer for helpful suggestions in preparing this new edition.

All additions and changes to the first edition were carefully implemented by Suzanne Kronenberg, using the Springer TeX-Macros package, and I gratefully acknowledge her help.

Zürich, June 1996 *Michael Struwe*

Preface to the First Edition

It would be hopeless to attempt to give a complete account of the history of the calculus of variations. The interest of Greek philosophers in isoperimetric problems underscores the importance of "optimal form" already in ancient cultures; see Hildebrandt-Tromba [1] for a beautiful treatise of this subject. While variational problems thus are part of our classical cultural heritage, the first modern treatment of a variational problem is attributed to Fermat, see Goldstine [1; p. 1]. Postulating that light follows a path of least possible time, in 1662 Fermat was able to derive the laws of refraction, thereby using methods which may already be termed analytic.

With the development of the Calculus by Newton and Leibniz, the basis was laid for a more systematic development of the calculus of variations. The brothers Johann and Jakob Bernoulli and Johann's student Leonhard Euler, all from the city of Basel in Switzerland, were to become the "founding fathers" (Hildebrandt-Tromba [1; p. 21]) of this new discipline. In 1743 Euler [1] submitted "A method for finding curves enjoying certain maximum or minimum properties", published in 1744, the first textbook on the calculus of variations. In an appendix to this book Euler [1; Appendix II, p. 298] expresses his belief that "every effect in nature follows a maximum or minimum rule" (see also Goldstine [1; p. 106]), a credo in the universality of the calculus of variations as a tool. The same conviction also shines through Maupertuis' [1] work on the famous "least action principle", also published in 1744. (In retrospect, however, it seems that Euler was the first to observe this important principle. See for instance Goldstine [1; p. 67 f. and p. 101 ff.] for a more detailed historical account.) Euler's book was a great source of inspiration for generations of mathematicians following.

Major contributions were made by Lagrange, Legendre, Jacobi, Clebsch, Mayer, and Hamilton to whom we owe what we now call "Euler-Lagrange equations", the "Jacobi differential equation" for a family of extremals, or "Hamilton-Jacobi theory".

The use of variational methods was not at all limited to one-dimensional problems in the mechanics of mass-points. In the 19^{th} century variational methods also were employed for instance to determine the distribution of an electrical charge on the surface of a conductor from the requirement that the energy of the associated electrical field be minimal ("Dirichlet's principle"; see Dirichlet [1] or Gauss [1]) or were used in the construction of analytic functions (Riemann [1]).

However, none of these applications was carried out with complete rigor. Often the model was confused with the phenomenon that it was supposed to describe and the fact (?) that for instance in nature there always exists an

equilibrium distribution for an electrical charge on a conducting surface was taken as sufficient evidence for the corresponding mathematical problem to have a solution. A typical reasoning reads as follows:

"In any event therefore the integral will be non-negative and hence there must exist a distribution (of charge) for which this integral assumes its minimum value," (Gauss [1; p. 232], translation by the author).

However, towards the end of the 19^{th} century progress in abstraction and a better understanding of the foundations of the calculus opened such arguments to criticism. Soon enough, Weierstrass [1; pp. 52–54] found an example of a variational problem that did not admit a minimum solution. Weierstrass challenged his colleagues to find a continuously differentiable function $u\colon [-1,1] \to \mathbb{R}$ minimizing the integral

$$I(u) = \int_{-1}^{1} \left| x \frac{d}{dx} u \right|^2 \, dx$$

subject (for instance) to the boundary conditions $u(\pm 1) = \pm 1$. Choosing

$$u_\varepsilon(x) = \frac{\arctan(\frac{x}{\varepsilon})}{\arctan(\frac{1}{\varepsilon})}, \ \varepsilon > 0,$$

as a family of comparison functions, Weierstrass was able to show that the infinium of I in the above class was 0; however, the value 0 is not attained. (See also Goldstine [1; p. 371 f.].) Weierstrass' critique of Dirichlet's principle precipitated the calculus of variations into a Grundlagenkrise comparable to the crisis in set theory and logic after Russel's discovery of antinomies in Cantor's set theory or Gödel's incompleteness proof.

However, through the combined efforts of several mathematicians who did not want to give up the wonderful tool that Dirichlet's principle had been – including Weierstrass, Arzéla, Fréchet, Hilbert, and Lebesgue – the calculus of variations was revalidated and emerged from its crisis with new strength and vigor.

Hilbert's speech at the centennial assembly of the International Congress 1900 in Paris, where he proposed his famous 20 problems – two of which were devoted to questions related to the calculus of variatons – marks this newly found confidence.

In fact, following Hilbert's [1] and Lebesgue's [1] solution of the Dirichlet problem, a development began which within a few decades brought tremendous success, highlighted by the 1929 theorem of Ljusternik and Schnirelman [1] on the existence of three distinct prime closed geodesics on any compact surface of genus zero, or the 1930/31 solution of Plateau's problem by Douglas [1], [2] and Radò [1].

The Ljusternik-Schnirelman result (and a previous result by Birkhoff [1], proving the existence of one closed geodesic on a surface of genus 0) also marks the beginning of global analyis. This goes beyond Dirichlet's principle as we no longer consider only minimizers (or maximizers) of variational

integrals, but instead look at all their critical points. The work of Ljusternik and Schnirelman revealed that much of the complexity of a function space is invariably reflected in the set of critical points of any variational integral defined on it, an idea whose importance for the further development of mathematics can hardly be overestimated, whose implications even today may only be conjectured, and whose applications seem to be virtually unlimited. Later, Ljusternik and Schnirelman [2] laid down the foundations of their method in a general theory. In honor of their pioneering effort any method which seeks to draw information concerning the number of critical points of a functional from topological data today often is referred to as Ljusternik-Schnirelman theory.

Around the time of Ljusternik and Schnirelman's work, another – equally important – approach towards a global theory of critical points was pursued by Marston Morse [2]. Morse's work also reveals a deep relation between the topology of a space and the number and types of critical points of any function defined on it. In particular, this led to the discovery of unstable minimal surfaces through the work of Morse-Tompkins [1], [2] and Shiffman [1], [2]. Somewhat reshaped and clarified, in the 50's Morse theory was highly successful in topology (see Milnor [1] and Smale [1]). After Palais [1], [2] and Smale [2] in the 60's succeeded in generalizing Milnor's constructions to infinite-dimensional Hilbert manifolds – see also Rothe [1] for some early work in this regard – Morse theory finally was recognized as a useful (and usable) instrument also for dealing with partial differential equations.

However, applications of Morse theory seemed somewhat limited in view of prohibitive regularity and non-degeneracy conditions to be met in a variational problem, conditions which – by the way – were absent in Morse's original work. Today, inspired by the deep work of Conley [1], Morse theory seems to be turning back to its origins again. In fact, a Morse-Conley theory is emerging which one day may provide a tool as universal as Ljusternik-Schnirelman theory and still offer an even better resolution of the relation between the critical set of a functional and topological properties of its domain. However, in spite of encouraging results, for instance by Benci [4], Conley-Zehnder [1], Jost-Struwe [1], Rybakowski [1], [2], Rybakowski-Zehnder [1], Salamon [1], and – in particular – Floer [1], a general theory of this kind does not yet exist.

In these notes we want to give an overview of the state of the art in some areas of the calculus of variations. Chapter I deals with the classical direct methods and some of their recent extensions. In Chapters II and III we discuss minimax methods, that is, Ljusternik-Schnirelman theory, with an emphasis on some limiting cases in the last chapter, leaving aside the issue of Morse theory whose face is currently changing all too rapidly.

Examples and applications are given to semilinear elliptic partial differential equations and systems, Hamiltonian systems, nonlinear wave equations, and problems related to harmonic maps of Riemannian manifolds or surfaces of prescribed mean curvature. Although our selection is of course biased by the interests of the author, an effort has been made to achieve a good balance between different areas of current research. Most of the results are known;

some of the proofs have been reworked and simplified. Attributions are made to the best of the author's knowledge. No attempt has been made to give an exhaustive account of the field or a complete survey of the literature.

General references for related material are Berger-Berger [1], Berger [1], Chow-Hale [1], Eells [1], Nirenberg [1], Rabinowitz [11], Schwartz [2], Zeidler [1]; in particular, we recommend the recent books by Ekeland [2] and Mawhin-Willem [1] on variational methods with a focus on Hamiltonian systems and the forthcoming works of Chang [7] and Giaquinta-Hildebrandt. Besides, we mention the classical textbooks by Krasnoselskii [1] (see also Krasnoselskii-Zabreiko [1]), Ljusternik-Schnirelman [2], Morse [2], and Vainberg [1]. As for applications to Hamiltonian systems and nonlinear variational problems, the interested reader may also find additional references on a special topic in these fields in the short surveys by Ambrosetti [2], Rabinowitz [9], or Zehnder [1].

The material covered in these notes is designed for advanced graduate or Ph.D. students or anyone who wishes to acquaint himself with variational methods and possesses a working knowledge of linear functional analysis and linear partial differential equations. Being familiar with the definitions and basic properties of Sobolev spaces as provided for instance in the book by Gilbarg-Trudinger [1] is recommended. However, some of these prerequisites can also be found in the appendix.

In preparing this manuscript I have received help and encouragement from a number of friends and colleagues. In particular, I wish to thank Proff. Herbert Amann and Hans-Wilhelm Alt for helpful comments concerning the first two sections of Chapter I. Likewise, I am indebted to Prof. Jürgen Moser for useful suggestions concerning Section I.4 and to Proff. Helmut Hofer and Eduard Zehnder for advice on Sections I.6, II.5, and II.8, concerning Hamiltonian systems.

Moreover, I am grateful to Gabi Hitz, Peter Bamert, Jochen Denzler, Martin Flucher, Frank Josellis, Thomas Kerler, Malte Schünemann, Miguel Sofer, Jean-Paul Theubet, and Thomas Wurms for going through a set of preliminary notes for this manuscript with me in a seminar at ETH Zürich during the winter term of 1988/89. The present text certainly has profited a great deal from their careful study and criticism.

Special thanks I also owe to Kai Jenni for the wonderful typesetting of this manuscript with the TEX text processing system.

I dedicate this book to my wife Anne.

Zürich, January 1990 *Michael Struwe*

Contents

Chapter I. The Direct Methods in the Calculus of Variations 1

1. Lower Semi-continuity 2

Degenerate Elliptic Equations, 4 — Minimal Partitioning Hypersurfaces, 6 — Minimal Hypersurfaces in Riemannian Manifolds, 7 — A General Lower Semi-continuity Result, 8

2. Constraints 13

Semilinear Elliptic Boundary Value Problems, 14 — Perron's Method in a Variational Guise, 16 — The Classical Plateau Problem, 19

3. Compensated Compactness 25

Applications in Elasticity, 29 — Convergence Results for Nonlinear Elliptic Equations, 32 — Hardy Space Methods, 35

4. The Concentration-Compactness Principle 36

Existence of Extremal Functions for Sobolev Embeddings, 42

5. Ekeland's Variational Principle 51

Existence of Minimizers for Quasi-convex Functionals, 54

6. Duality 58

Hamiltonian Systems, 60 — Periodic Solutions of Nonlinear Wave Equations, 65

7. Minimization Problems Depending on Parameters 69

Harmonic Maps with Singularities, 71

Chapter II. Minimax Methods 74

1. The Finite Dimensional Case 74

2. The Palais-Smale Condition 77

3. A General Deformation Lemma 81

Pseudo-gradient Flows on Banach Spaces, 81 — Pseudo-gradient Flows on Manifolds, 85

4. The Minimax Principle 87

Closed Geodesics on Spheres, 89

5. Index Theory 94

Krasnoselskii Genus, 94 — Minimax Principles for Even Functionals, 96 — Applications to Semilinear Elliptic Problems, 98 — General Index Theories, 99 — Ljusternik-Schnirelman Category, 100 — A Geometrical S^1-Index, 101 — Multiple Periodic Orbits of Hamiltonian Systems, 103

6. The Mountain Pass Lemma and its Variants 108

Applications to Semilinear Elliptic Boundary Value Problems, 110 — The Symmetric Mountain Pass Lemma, 112 — Application to Semilinear Equations with Symmetry, 116

7. Perturbation Theory 118

Applications to Semilinear Elliptic Equations, 120

8. Linking 125

Applications to Semilinear Elliptic Equations, 128 — Applications to Hamiltonian Systems, 130

9. Parameter Dependence 137

10. Critical Points of Mountain Pass Type 143

Multiple Solutions of Coercive Elliptic Problems, 147

11. Non-differentiable Functionals 150

12. Ljusternik-Schnirelman Theory on Convex Sets 162

Applications to Semilinear Elliptic Boundary Value Problems, 166

Chapter III. Limit Cases of the Palais-Smale Condition 169

1. Pohožaev's Non-existence Result 170

2. The Brezis-Nirenberg Result 173

Constrained Minimization, 174 — The Unconstrained Case: Local Compactness, 175 — Multiple Solutions, 180

3. The Effect of Topology 183

A Global Compactness Result, 184 — Positive Solutions on Annular-Shaped Regions, 190

4. The Yamabe Problem 194

The Variational Approach, 195 — The Locally Conformally Flat Case, 197 — The Yamabe Flow, 198 — The Proof of Theorem 4.9 (following Ye [1]), 200 — Convergence of the Yamabe Flow in the General Case, 204 — The Compact Case $u_\infty > 0$, 211 — Bubbling: The Case $u_\infty \equiv 0$, 216

5. The Dirichlet Problem for the Equation of Constant Mean Curvature 220

Small Solutions, 221 — The Volume Functional, 223 — Wente's Uniqueness Result, 225 — Local Compactness, 226 — Large Solutions, 229

6. Harmonic Maps of Riemannian Surfaces 231

The Euler-Lagrange Equations for Harmonic Maps, 232 — Bochner identity, 234 — The Homotopy Problem and its Functional Analytic Setting, 234 — Existence and Non-existence Results, 237 — The Heat Flow for Harmonic Maps, 238 — The Global Existence Result, 239 — The Proof of Theorem 6.6, 242 — Finite-Time Blow-Up, 253 — Reverse Bubbling and Nonuniqueness, 257

Appendix A 263

Sobolev Spaces, 263 — Hölder Spaces, 264 — Imbedding Theorems, 264 — Density Theorem, 265 — Trace and Extension Theorems, 265 — Poincaré Inequality, 266

Appendix B 268

Schauder Estimates, 268 — L^p-Theory, 268 — Weak Solutions, 269 — A Regularity Result, 269 — Maximum Principle, 271 — Weak Maximum Principle, 272 — Application, 273

Appendix C 274

Fréchet Differentiability, 274 — Natural Growth Conditions, 276

References 277

Index 301

Glossary of Notations

V, V^*	generic Banach space with dual V^*
$\|\cdot\|$	norm in V
$\|\cdot\|_*$	induced norm in V^*, often also denoted $\|\cdot\|$
$\langle\cdot,\cdot\rangle\colon V\times V^*\to\mathbb{R}$	dual pairing, occasionally also used to denote scalar product in $\mathbb{R}^n$
E	generic energy functional
DE	Fréchet derivative
$\mathrm{Dom}(E)$	domain of E
$\langle v, DE(u)\rangle = DE(u)v = D_vE(u)$	directional derivative of E at u in direction v
$L^p(\Omega;\mathbb{R}^n)$	space of Lebesgue-measurable functions $u\colon\Omega\to\mathbb{R}^n$ with finite L^p-norm $\|u\|_{L^p} = \big(\int_\Omega \|u\|^p\,dx\big)^{1/p},\ 1\le p<\infty$
$L^\infty(\Omega;\mathbb{R}^n)$	space of Lebesgue-measurable and essentially bounded functions $u\colon\Omega\to\mathbb{R}^n$ with norm $\|u\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\Omega}\|u(x)\|$
$H^{m,p}(\Omega;\mathbb{R}^n)$	Sobolev space of functions $u\in L^p(\Omega;\mathbb{R}^n)$ with $\|\nabla^k u\|\in L^p(\Omega)$ for all $k\in\mathbb{N}_0^n, \|k\|\le m$, with norm $\|u\|_{H^{m,p}} = \sum_{0\le\|k\|\le m}\|\nabla^k u\|_{L^p}$
$H_0^{m,p}(\Omega;\mathbb{R}^n)$	completion of $C_0^\infty(\Omega;\mathbb{R}^n)$ in the norm $\|\cdot\|_{H^{m,p}}$; if Ω is bounded an equivalent norm is given by $\|u\|_{H_0^{m,p}} = \sum_{\|k\|=m}\|\nabla^k u\|_{L^p}$
$H^{-m,q}(\Omega;\mathbb{R}^n)$	dual of $H_0^{m,p}(\Omega;\mathbb{R}^n)$, where $\frac{1}{p}=\frac{1}{q}=1$; q is omitted, if $p=q=2$
$D^{m,p}(\Omega;\mathbb{R}^n)$	completion of $C_0^\infty(\Omega;\mathbb{R}^n)$ in the norm $\|u\|_{D^{m,p}} = \sum_{\|k\|=m}\|\nabla^k u\|_{L^p}$

$C^{m,\alpha}(\Omega;\mathbb{R}^n)$	space of m times continuously differentiable functions $u\colon \Omega \to \mathbb{R}^n$ whose mth order derivatives are Hölder continuous with exponent $0 \le \alpha \le 1$
$C_0^\infty(\Omega;\mathbb{R}^n)$	space of smooth functions $u\colon \Omega \to \mathbb{R}^n$ with compact support in Ω
$\operatorname{supp}(u) = \overline{\{x \in \Omega\ ;\ u(x) \neq 0\}}$	support of a function $u\colon \Omega \to \mathbb{R}^n$
$\Omega' \subset\subset \Omega$	the closure of Ω' is compact and contained in Ω
$\llcorner$	restriction of a measure
$\mathcal{L}^n$	Lebesgue measure on $\mathbb{R}^n$
$B_\rho(u;V) = \{v \in V\ ;\ \|u-v\| < \rho\}$	open ball of radius ρ around $u \in V$; in particular, if $V = \mathbb{R}^n$, then $B_\rho(x_0) = B_\rho(x_0;\mathbb{R}^n)$, $B_\rho = B_\rho(0)$
Re	real part
Im	imaginary part
c, C	generic constants
Cross-references	$(N.x.y)$ refers to formula (x,y) in Chapter N $(x.y)$ within Chapter N refers to formula $(N.x.y)$

Chapter I

The Direct Methods in the Calculus of Variations

Many problems in analysis can be cast into the form of functional equations $F(u) = 0$, the solution u being sought among a class of admissible functions belonging to some Banach space V.

Typically, these equations are nonlinear; for instance, if the class of admissible functions is restricted by some (nonlinear) constraint.

A particular class of functional equations is the class of Euler-Lagrange equations

$$DE(u) = 0$$

for a functional E on V, which is Fréchet-differentiable with derivative DE. We say such equations are of *variational form.*

For equations of variational form an extensive theory has been developed, and variational principles play an important role in mathematical physics and differential geometry, optimal control and numerical anlysis.

We briefly recall the basic definitions that will be needed in this and the following chapters, see Appendix C for details: Suppose E is a Fréchet-differentiable functional on a Banach space V with normed dual V^* and duality pairing $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$, and let $DE : V \to V^*$ denote the Fréchet-derivative of E. Then the directional (Gateaux-) derivative of E at u in the direction of v is given by

$$\frac{d}{d\varepsilon} E(u + \varepsilon v)\Big|_{\varepsilon = 0} = \langle v, DE(u) \rangle = DE(u)\, v.$$

For such E, we call a point $u \in V$ *critical* if $DE(u) = 0$; otherwise, u is called *regular.* A number $\beta \in \mathbb{R}$ is a *critical value* of E if there exists a critical point u of E with $E(u) = \beta$. Otherwise, β is called *regular.* Of particular interest (also in the non-differentiable case) will be relative minima of E, possibly subject to constraints. Recall that for a set $M \subset V$ a point $u \in M$ is an *absolute minimizer* for E on M if for all $v \in M$ there holds $E(v) \geq E(u)$. A point $u \in M$ is a *relative minimizer* for E on M if for some neighborhood U of u in V it is absolutely E-minimizing in $M \cap U$. Moreover, in the differentiable case, we shall also be interested in the existence of *saddle points*, that is, critical points u of E such that any neighborhood U of u in V contains points v, w such that $E(v) < E(u) < E(w)$. In physical systems, saddle points appear as unstable equilibria or transient excited states.

In this chapter we review some basic methods for proving the existence of relative minimizers. Somewhat imprecisely we summarily refer to these methods as the *direct methods* in the calculus of variations. However, besides the classical lower semi-continuity and compactness method we also include the compensated compactness method of Murat and Tartar, and the concentration-compactness principle of P.L. Lions. Moreover, we recall Ekeland's variational principle and the duality method of Clarke and Ekeland.

Applications will be given to problems concerning minimal hypersurfaces, semilinear and quasi-linear elliptic boundary value problems, finite elasticity, Hamiltonian systems, and semilinear wave equations.

From the beginning it will be apparent that in order to achieve a satisfactory existence theory the notion of solution will have to be suitably relaxed. Hence, in general, the above methods will at first only yield generalized or "weak" solutions of our problems. A second step often will be necessary to show that these solutions are regular enough to be admitted as classical solutions. The regularity theory in many cases is very subtle and involves a delicate machinery. It would go beyond the scope of this book to cover this topic completely. However, for the problems that we will mostly be interested in, the regularity question can be dealt with rather easily. The reader will find this material in Appendix B. References to more advanced texts on the regularity issue will be given where appropriate.

1. Lower Semi-continuity

In this section we give sufficient conditions for a functional to be bounded from below and to attain its infimum.

The discussion can be made largely independent of any differentiability assumptions on E or structure assumptions on the underlying space of admissible functions M. In fact, we have the following classical result.

1.1 Theorem. *Let M be a topological Hausdorff space, and suppose $E : M \to \mathbb{R} \cup +\infty$ satisfies the condition of bounded compactness:*

$$\begin{aligned} &\textit{For any } \alpha \in \mathbb{R} \textit{ the set} \\ &\qquad K_\alpha = \{u \in M \ ; \ E(u) \leq \alpha\} \\ &\textit{is compact (Heine-Borel property).} \end{aligned} \tag{1.1}$$

Then E is uniformly bounded from below on M and attains its infimum. The conclusion remains valid if instead of (1.1) we suppose that any sub-level set K_α is sequentially compact.

Remark. Necessity of condition (1.1) is illustrated by simple examples: The function $E\colon [-1,1] \to \mathbb{R}$ given by $E(x) = x^2$ if $x \neq 0$, $E(x) = 1$ if $x = 0$, or the exponential function $E(x) = \exp(x)$ on $\mathbb{R}$ are bounded from below but do not admit a minimizer. Note that the space M in the first example is compact while in the second example the function E is smooth – even analytic.

Proof of Theorem 1.1. Suppose (1.1) holds. We may assume $E \not\equiv +\infty$. Let

$$\alpha_0 = \inf_M E \geq -\infty,$$

and let (α_m) be the strictly decreasing sequence

$$\alpha_m \searrow \alpha_0 \qquad (m \to \infty)\ .$$

Let $K_m = K_{\alpha_m}$. By assumption, each K_m is compact and non-empty. Moreover, $K_m \supset K_{m+1}$ for all m. By compactness of K_m there exists a point $u \in \bigcap_{m\in\mathbb{N}} K_m$, satisfying

$$E(u) \leq \alpha_m, \qquad \text{for all } m.$$

Passing to the limit $m \to \infty$ we obtain that

$$E(u) \leq \alpha_0 = \inf_M E,$$

and the claim follows.

If instead of (1.1) each K_α is sequentially compact, we choose a *minimizing sequence* (u_m) in M such that $E(u_m) \to \alpha_0$. Then for any $\alpha > \alpha_0$ the sequence (u_m) will eventually lie entirely within K_α. By sequential compactness of K_α therefore (u_m) will accumulate at a point $u \in \bigcap_{\alpha>\alpha_0} K_\alpha$ which is the desired minimizer. □

Note that if $E : M \to \mathbb{R}$ satisfies (1.1), then for any $\alpha \in \mathbb{R}$ the set

$$\{u \in M\ ;\ E(u) > \alpha\} = M \setminus K_\alpha$$

is open, that is, E is *lower semi-continuous.* (Respectively, if each K_α is sequentially compact, then E will be sequentially lower semi-continuous.) Conversely, if E is (sequentially) lower semi-continuous and for some $\overline{\alpha} \in \mathbb{R}$ the set $K_{\overline{\alpha}}$ is (sequentially) compact, then K_α will be (sequentially) compact for all $\alpha \leq \overline{\alpha}$ and again the conclusion of Theorem 1.1 will be valid.

Note that the lower semi-continuity condition can be more easily fulfilled the finer the topology on M. In contrast, the condition of compactness of the sub-level sets K_α , $\alpha \in \mathbb{R}$, calls for a coarse topology and both conditions are competing. In practice, there is often a natural weak Sobolev space topology where both conditions can be simultaneously satisfied. However, there are many interesting cases where condition (1.1) cannot hold in *any* reasonable topology (even though relative minimizers may exist). Later in this chapter we

shall see some examples and some more delicate ways of handling the possible loss of compactness. See Section 4; see also Chapter III.

In applications, the conditions of the following special case of Theorem 1.1 can often be checked more easily.

1.2 Theorem. *Suppose V is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset V$ be a weakly closed subset of V. Suppose $E : M \to \mathbb{R} \cup +\infty$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to V, that is, suppose the following conditions are fullfilled:*
(1°) $E(u) \to \infty$ as $\|u\| \to \infty$, $u \in M$.
(2°) For any $u \in M$, any sequence (u_m) in M such that $u_m \rightharpoonup u$ weakly in V there holds:

$$E(u) \leq \liminf_{m\to\infty} E(u_m) \ .$$

Then E is bounded from below on M and attains its infimum in M.

The concept of minimizing sequences offers a direct and (apparently) constructive proof.

Proof. Let $\alpha_0 = \inf_M E$ and let (u_m) be a minimizing sequence in M, that is, satisfying $E(u_m) \to \alpha_0$. By coerciveness, (u_m) is bounded in V. Since V is reflexive, by the Eberlein-Šmulian theorem (see Dunford-Schwartz [1; p. 430]) we may assume that $u_m \rightharpoonup u$ weakly for some $u \in V$. But M is weakly closed, therefore $u \in M$, and by weak lower semi-continuity

$$E(u) \leq \liminf_{m\to\infty} E(u_m) = \alpha_0 \ . \qquad \square$$

Examples. An important example of a sequentially weakly lower semicontinous functional is the norm in a Banach space V. Closed and convex subsets of Banach spaces are important examples of weakly closed sets. If V is the dual of a separable normed vector space, Theorem 1.2 and its proof remain valid if we replace weak by weak*-convergence.

We present some simple applications.

Degenerate Elliptic Equations

1.3 Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$, $p \in [2, \infty[$ with conjugate exponent q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and let $f \in H^{-1,q}(\Omega)$, the dual of $H_0^{1,p}(\Omega)$, be given. Then there exists a weak solution $u \in H_0^{1,p}(\Omega)$ of the boundary value problem*

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega \tag{1.2}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{1.3}$$

in the sense that u satisfies the equation

$$\int_\Omega (\nabla u|\nabla u|^{p-2}\nabla\varphi - f\varphi)dx = 0\ , \qquad \forall\varphi \in C_0^\infty(\Omega)\ . \tag{1.4}$$

Proof. Note that the left part of (1.4) is the directional derivative of the C^1-functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx \ - \ \int_\Omega fu\,dx$$

on the Banach space $V = H_0^{1,p}(\Omega)$ in direction φ; that is, problem (1.2), (1.3) is of variational form.

Note that $H_0^{1,p}(\Omega)$ is reflexive. Moreover, E is coercive. In fact, we have

$$\begin{aligned} E(u) &\geq \frac{1}{p}\|u\|^p_{H_0^{1,p}} \ - \ \|f\|_{H^{-1,q}}\ \|u\|_{H_0^{1,p}} \geq \frac{1}{p}\left(\|u\|^p_{H_0^{1,p}} - c\|u\|_{H_0^{1,p}}\right) \\ &\geq c^{-1}\|u\|^p_{H_0^{1,p}} \ - \ C. \end{aligned}$$

Finally, E is (sequentially) weakly lower semi-continuous: It suffices to show that for $u_m \rightharpoonup u$ weakly in $H_0^{1,p}(\Omega)$ we have

$$\int_\Omega f\,u_m\,dx \ \ \to \ \ \int_\Omega f\,u\,dx\ .$$

Since $f \in H^{-1,q}(\Omega)$, however, this follows from the very definition of weak convergence. Hence Theorem 1.2 is applicable and there exists a minimizer $u \in H_0^{1,p}(\Omega)$ of E, solving (1.4). □

Note that for $p \geq 2$ the p-Laplacian is strongly monotone in the sense that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot(\nabla u - \nabla v)\,dx \geq c\|u-v\|^p_{H_0^{1,p}}\ .$$

In particular, the solution u to (1.4) is unique.

If f is more regular, say $f \in C^{m,\alpha}(\overline{\Omega})$, we would expect the solution u of (1.4) to be more regular as well. This is true if $p = 2$, see Appendix B, but in the degenerate case $p > 2$, where the uniform ellipticity of the p-Laplace operator is lost at zeros of $|\nabla u|$, the best that one can hope for is $u \in C^{1,\alpha}(\overline{\Omega})$; see Uhlenbeck [1], Tolksdorf [2; p. 128], Di Benedetto [1].

In Theorem 1.3 we have applied Theorem 1.2 to a functional on a reflexive space. An example in a non-reflexive setting is given next.

Minimal Partitioning Hypersurfaces

For a domain $\Omega \subset \mathbb{R}^n$ let $BV(\Omega)$ be the space of functions $u \in L^1(\Omega)$ such that

$$\int_\Omega |Du| = \sup\Big\{ \int_\Omega \sum_{i=1}^n u D_i g_i \, dx \; ; \\ g = (g_1, \dots, g_n) \in C_0^1(\Omega; \mathbb{R}^n), \; |g| \le 1 \Big\} < \infty \; ,$$

endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + \int_\Omega |Du| \; .$$

$BV(\Omega)$ is a Banach space, embedded in $L^1(\Omega)$, and – provided Ω is bounded and sufficiently smooth – by Rellich's theorem the injection $BV(\Omega) \hookrightarrow L^1(\Omega)$ is compact; see for instance Giusti [1; Theorem 1.19, p. 17]. Moreover, the function $u \mapsto \int_\Omega |Du|$ is lower semi-continuous with respect to L^1-convergence.

Let χ_G be the characteristic function of a set $G \subset \mathbb{R}^n$; that is, $\chi_G(x) = 1$ if $x \in G$, $\chi_G(x) = 0$ else. Also let $\mathcal{L}^n$ denote the n-dimensional Lebesgue measure.

1.4 Theorem. *Let Ω be a smooth, bounded domain in $\mathbb{R}^n$. Then there exists a subset $G \subset \Omega$ such that*

(1°) $$\mathcal{L}^n(G) = \mathcal{L}^n(\Omega \setminus G) = \frac{1}{2}\mathcal{L}^n(\Omega)$$

and such that its perimeter with respect to Ω,

(2°) $$P(G, \Omega) = \int_\Omega |D\chi_G| \, ,$$

is minimal among all sets satisfying (1°).

Proof. Let $M = \{\chi_G \; ; \; G \subset \Omega$ is measurable and satisfies $(1°)\}$, endowed with the L^1-topology, and let $E : M \to \mathbb{R} \cup +\infty$ be given by

$$E(u) = \int_\Omega |Du| \; .$$

Since $\|\chi_G\|_{L^1} \le \mathcal{L}^n(\Omega)$, the functional E is coercive on M with respect to the norm in $BV(\Omega)$. Since bounded sets in $BV(\Omega)$ are relatively compact in $L^1(\Omega)$ and since M is closed in $L^1(\Omega)$, by weak lower semi-continuity of E in $L^1(\Omega)$ the sub-level sets of E are compact. The conclusion now follows from Theorem 1.1. □

The support of the distribution $D\chi_G$, where G has minimal perimeter $(2°)$ with respect to Ω, can be interpreted as a minimal bisecting hypersurface,

dividing Ω into two regions of equal volume. The regularity of the dividing hypersurface is intimately connected with the existence of minimal cones in $\mathbb{R}^n$. See Giusti [1] for further material on functions of bounded variation, sets of bounded perimeter, the area integrand, and applications.

A related setting for the study of minimal hypersurfaces and related objects is offered by geometric measure theory. Also in this field variational principles play an important role; see for instance Almgren [1], Morgan [1], or Simon [1] for introductory material and further references.

Our next example is concerned with a parametric approach.

Minimal Hypersurfaces in Riemannian Manifolds

Let Ω be a bounded domain in $\mathbb{R}^n$, and let S be a compact subset in $\mathbb{R}^N$. Also let $u_0 \in H^{1,2}(\Omega;\mathbb{R}^N)$ with $u_0(\Omega) \subset S$ be given. Define

$$H^{1,2}(\Omega;S) = \left\{u \in H^{1,2}(\Omega;\mathbb{R}^N)\ ;\ u(\Omega) \subset S \ \text{ almost everywhere}\right\}$$

and let

$$M = \left\{u \in H^{1,2}(\Omega;S)\ ;\ u - u_0 \in H_0^{1,2}(\Omega;\mathbb{R}^N)\right\}\ .$$

Then, by Rellich's theorem, M is closed in the weak topology of $V = H^{1,2}(\Omega;\mathbb{R}^N)$. For $u = (u^1,\dots,u^N) \in H^{1,2}(\Omega;S)$ let

$$E(u) = \int_\Omega g_{ij}(u)\nabla u^i \nabla u^j\ dx\ ,$$

where $g = (g_{ij})_{1\le i,j\le N}$ is a given positively definite symmetric matrix with coefficients $g_{ij}(u)$ depending continuously on $u \in S$, and where, by convention, we tacitly sum over repeated indices $1 \le i,j \le N$. Note that since S is compact g is uniformly positive definite on S, and there exists $\lambda > 0$ such that $E(u) \ge \lambda \|\nabla u\|_{L^2}^2$ for $u \in H^{1,2}(\Omega;S)$. In addition, since S and Ω are bounded, we have that $\|u\|_{L^2} \le c$ uniformly, for $u \in H^{1,2}(\Omega;S)$. Hence E is coercive on $H^{1,2}(\Omega;S)$ with respect to the norm in $H^{1,2}(\Omega;\mathbb{R}^N)$.

Finally, E is lower semi-continuous in $H^{1,2}(\Omega;S)$ with respect to weak convergence in $H^{1,2}(\Omega;\mathbb{R}^N)$. Indeed, if $u_m \rightharpoonup u$ weakly in $H^{1,2}(\Omega;\mathbb{R}^N)$, by Rellich's theorem $u_m \to u$ strongly in L^2 and hence a subsequence (u_m) converges almost everywhere. By Egorov's theorem, given $\delta > 0$ there is an exceptional set Ω_δ of measure $\mathcal{L}^n(\Omega_\delta) < \delta$ such that $u_m \to u$ uniformly on $\Omega \setminus \Omega_\delta$. We may assume that $\Omega_\delta \subset \Omega_{\delta'}$ for $\delta \le \delta'$. By weak lower semi-continuity of the semi-norm on $H^{1,2}(\Omega;\mathbb{R}^N)$, defined by

$$|v|^2 = \int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\ \nabla v^i \nabla v^j\ dx,$$

then

$$\begin{aligned}\int_{\Omega\setminus\Omega_\delta} & g_{ij}(u)\,\nabla u^i \nabla u^j\,dx \\ &\le \liminf_{m\to\infty} \int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\,\nabla u_m^i \nabla u_m^j\,dx \\ &= \liminf_{m\to\infty} \int_{\Omega\setminus\Omega_\delta} g_{ij}(u_m)\,\nabla u_m^i \nabla u_m^j\,dx \\ &\le \liminf_{m\to\infty} E(u_m)\ .\end{aligned}$$

Passing to the limit $\delta \to 0$, from Beppo Levi's theorem we obtain

$$\begin{aligned}E(u) &= \lim_{\delta\to 0} \int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\,\nabla u^i \nabla u^j\,dx \\ &\le \liminf_{m\to\infty}\ E(u_m)\ .\end{aligned}$$

Applying Theorem 1.2 to E on M we obtain

1.5 Theorem. *For any boundary data $u_0 \in H^{1,2}(\Omega; S)$ there exists an E-minimal extension $u \in M$.*

In differential geometry Theorem 1.5 arises in the study of harmonic maps $u : \Omega \to S$ from a domain Ω into an N-dimensional manifold S with metric g for prescribed boundary data $u = u_0$ on $\partial\Omega$. Like in the previous example, the regularity question is related to the existence of special harmonic maps; in this case, singularities of harmonic maps from Ω into S are related to harmonic mappings of spheres into S. For further references see Eells-Lemaire [1], [2], Hildebrandt [3], Jost [2]. For questions concerning regularity see Giaquinta-Giusti [1], Schoen-Uhlenbeck [1], [2].

A General Lower Semi-continuity Result

We now conclude this short list of introductory examples and return to the development of the variational theory. Note that the property of E being lower semi-continuous with respect to some weak kind of convergence is at the core of the above existence results. In Theorem 1.6 below we establish a lower semi-continuity result for a very broad class of variational integrals, including and going beyond those encountered in Theorem 1.5, as Theorem 1.6 would also apply in the case of unbounded targets S and possibly degenerate or singular metrics g.

We consider variational integrals

$$E(u) = \int_\Omega F(x, u, \nabla u)\,dx \tag{1.5}$$

involving (vector-valued) functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$.

1.6 Theorem. *Let Ω be a domain in $\mathbb{R}^n$, and assume that $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Caratheodory function satisfying the conditions*
(1°) $F(x,u,p) \geq \phi(x)$ for almost every x,u,p, where $\phi \in L^1(\Omega)$.
(2°) $F(x,u,\cdot)$ is convex in p for almost every x,u.
Then, if $u_m,\ u \in H^{1,1}_{loc}(\Omega)$ and $u_m \to u$ in $L^1(\Omega')$, $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^1(\Omega')$ for all bounded $\Omega' \subset\subset \Omega$, it follows that

$$E(u) \leq \liminf_{m\to\infty} E(u_m)\ ,$$

where E is given by (1.5).

Notes. In the scalar case $N = 1$, weak lower semi-continuity results like Theorem 1.6 were first stated by Tonelli [1] and Morrey [1]; these results were then extended and simplified by Serrin [1], [2] who showed that for non-negative, smooth functions $F(x,u,p)\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, which are convex in p , the functional E given by (1.5) is lower semi-continuous with respect to convergence in $L^1_{loc}(\Omega)$. A corresponding result in the vector-valued case $N > 1$ subsequently was derived by Morrey [4; Theorem 4.1.1]; however, Eisen [1] not only pointed out a gap in Morrey's proof but also gave an example showing that for $N > 1$ in general, Theorem 1.6 ceases to be true without the assumption that the L^1-norms of ∇u_m are uniformly locally bounded. Theorem 1.6 is due to Berkowitz [1] and Eisen [2]. Related results can be found for instance in Morrey [4; Theorem 1.8.2], or Giaquinta [1]. Our proof is modeled on Eisen [2].

Proof. We may assume that $\big(E(u_m)\big)$ is finite and convergent. Moreover, replacing F by $F - \phi$ we may assume that $F \geq 0$. Let $\Omega' \subset\subset \Omega$ be given. By weak local L^1-convergence $\nabla u_m \rightharpoonup \nabla u$, for any $m_0 \in \mathbb{N}$ there exists a sequence $(P^l)_{l\geq m_0}$ of convex linear combinations

$$P^l = \sum_{m=m_0}^{l} \alpha_m^l \nabla u_m\ ,\ 0 \leq \alpha_m^l \leq 1\ ,\ \sum_{m=m_0}^{l} \alpha_m^l = 1\ ,\ l \geq m_0$$

such that $P^l \to \nabla u$ strongly in $L^1(\Omega')$ and pointwise almost everywhere as $l \to \infty$; see for instance Rudin [1; Theorem 3.13]. By convexity, for any m_0, any $l \geq m_0$, and almost every $x \in \Omega'$:

$$\begin{aligned} F\left(x, u(x), P^l(x)\right) &= F\left(x, u(x), \sum_{m=m_0}^{l} \alpha_m^l \nabla u_m(x)\right) \\ &\leq \sum_{m=m_0}^{l} \alpha_m^l F\left(x, u(x), \nabla u_m(x)\right)\ . \end{aligned}$$

Integrating over Ω' and passing to the limit $l \to \infty$, from Fatou's lemma we obtain:

$$\int_{\Omega'} F\left(x, u(x), \nabla u(x)\right)\, dx \leq \liminf_{l\to\infty} \int_{\Omega'} F\left(x, u(x), P^l(x)\right)\, dx$$
$$\leq \sup_{m\geq m_0} \int_{\Omega'} F\big(x, u(x), \nabla u_m(x)\big)\, dx\ .$$

Since m_0 was arbitrary, this implies that

$$\int_{\Omega'} F\big(x, u(x), \nabla u(x)\big)\, dx \leq \limsup_{m\to\infty} \int_{\Omega'} F\left(x, u(x), \nabla u_m(x)\right)\, dx\,,$$

for any bounded $\Omega' \subset\subset \Omega$.

Now we need the following result (Eisen [2; p. 75]).

1.7 Lemma. *Under the hypotheses of Theorem 1.6 on F, u_m, and u there exists a subsequence (u_m) such that:*

$$F\left(x, u_m(x), \nabla u_m(x)\right) - F\left(x, u(x), \nabla u_m(x)\right) \to 0$$

in measure, locally in Ω.

Proof of Theorem 1.6 (completed). By Lemma 1.7 for any $\Omega' \subset\subset \Omega$, any $\varepsilon > 0$, and any $m_0 \in \mathbb{N}$ there exists $m \geq m_0$ and a set $\Omega'_{\varepsilon,m} \subset \Omega'$ with $\mathcal{L}^n\left(\Omega'_{\varepsilon,m}\right) < \varepsilon$ such that

$$|F\big(x, u_m(x), \nabla u_m(x)\big) - F\big(x, u(x), \nabla u_m(x)\big)| < \varepsilon \tag{1.6}$$

for all $x \in \Omega' \setminus \Omega'_{\varepsilon,m}$. Replacing ε by $\varepsilon_m = 2^{-m}$ and passing to a subsequence, if necessary, we may assume that for each m there is a set $\Omega'_{\varepsilon_m,m} \subset \Omega'$ of measure $< \varepsilon_m$ such that (1.6) is satisfied (with ε_m) for all $x \in \Omega' \setminus \Omega'_{\varepsilon_m,m}$. Hence, for any given $\varepsilon > 0$, if we choose $m_0 = m_0(\varepsilon) > |\log_2 \varepsilon|$, $\Omega'_\varepsilon = \bigcup_{m\geq m_0} \Omega'_{\varepsilon_m,m}$, this set has measure $\mathcal{L}^n(\Omega'_\varepsilon) < \varepsilon$ and inequality (1.6) holds uniformly for all $x \in \Omega' \setminus \Omega'_\varepsilon$, and all $m \geq m_0(\varepsilon)$. Moreover, for $\varepsilon < \delta$ by construction $\Omega'_\varepsilon \subset \Omega'_\delta$.

Cover Ω by disjoint bounded sets $\Omega^{(k)} \subset\subset \Omega$, $k \in \mathbb{N}$. Let $\varepsilon > 0$ be given and choose a sequence $\varepsilon^{(k)} > 0$, such that $\sum_{k\in\mathbb{N}} \mathcal{L}^n\left(\Omega^{(k)}\right)\varepsilon^{(k)} \leq \varepsilon$. Passing to a subsequence, if necessary, for each $\Omega^{(k)}$ and $\varepsilon^{(k)}$ we may choose $m_0^{(k)}$ and $\Omega_\varepsilon^{(k)} \subset \Omega^{(k)}$ such that $\mathcal{L}^n\left(\Omega_\varepsilon^{(k)}\right) < \varepsilon^{(k)}$ and

$$|F\left(x, u_m(x), \nabla u_m(x)\right) - F\left(x, u(x), \nabla u_m(x)\right)| < \varepsilon^{(k)}$$

uniformly for $x \in \Omega^{(k)} \setminus \Omega_\varepsilon^{(k)}$, $m \geq m_0^{(k)}$. Moreover, we may assume that $\Omega_\varepsilon^{(k)} \subset \Omega_\delta^{(k)}$, if $\varepsilon < \delta$, for all k. Then for any $K \in \mathbb{N}$, letting $\Omega^K = \cup_{k=1}^K \Omega^{(k)}$, $\Omega_\varepsilon^K = \cup_{k=1}^K \Omega_\varepsilon^{(k)}$, we have

$$\int_{\Omega^K \setminus \Omega_\varepsilon^K} F(x,u,\nabla u)\,dx$$

$$\leq \limsup_{m\to\infty} \int_{\Omega^K \setminus \Omega_\varepsilon^K} F\left(x,u,\nabla u_m\right)\,dx$$

$$\leq \limsup_{m\to\infty} \int_{\Omega^K \setminus \Omega_\varepsilon^K} F\left(x,u_m,\nabla u_m\right)\,dx + \varepsilon$$

$$\leq \limsup_{m\to\infty} E(u_m) + \varepsilon = \liminf_{m\to\infty} E(u_m) + \varepsilon\ .$$

Letting $\varepsilon \to 0$ and then $K \to \infty$, the claim follows from Beppo Levi's theorem, since $F \geq 0$ and since $\Omega^K \setminus \Omega_\varepsilon^K$ increases when $\varepsilon \downarrow 0$, followed by $K \uparrow \infty$. □

Proof of Lemma 1.7. We basically follow Eisen [2]. Suppose by contradiction that there exist $\Omega' \subset\subset \Omega$ and $\varepsilon > 0$ such that, letting

$$\Omega_m = \{x \in \Omega'\ ;\ |F\left(x,u_m,\nabla u_m\right) - F\left(x,u,\nabla u_m\right)| \geq \varepsilon\}\ ,$$

there holds

$$\liminf_{m\to\infty} \mathcal{L}^n(\Omega_m) \geq 2\varepsilon\ .$$

The sequence (∇u_m), being weakly convergent, is uniformly bounded in $L^1(\Omega')$. In particular,

$$\mathcal{L}^n\{x \in \Omega'\ ;\ |\nabla u_m(x)| \geq l\} \leq l^{-1} \int_{\Omega'} |\nabla u_m|\,dx \leq \frac{C}{l} \leq \varepsilon\ ,$$

if $l \geq l_0(\varepsilon)$ is large enough. Setting $\tilde{\Omega}_m := \{x \in \Omega_m\ ;\ |\nabla u_m(x)| \leq l_0(\varepsilon)\}$ therefore there holds

$$\liminf_{m\to\infty} \mathcal{L}^n\left(\tilde{\Omega}_m\right) \geq \varepsilon.$$

Hence also for $\Omega^M = \bigcup_{m\geq M} \tilde{\Omega}_m$ we have

$$\mathcal{L}^n(\Omega^M) \geq \varepsilon\ ,$$

uniformly in $M \in \mathbb{N}$. Moreover, $\Omega' \supset \Omega^M \supset \Omega^{M+1}$ for all M and therefore $\Omega^\infty := \bigcap_{M\in\mathbb{N}} \Omega^M \subset \Omega'$ has $\mathcal{L}^n(\Omega^\infty) \geq \varepsilon$. Finally, neglecting a set of measure zero and passing to a subsequence, if necessary, we may assume that $F(x,z,p)$ is continuous in (z,p), that $u_m(x)$, $u(x)$, $\nabla u_m(x)$ are unambiguously defined and finite while $u_m(x) \to u(x)$ as $m \to \infty$ at every point $x \in \Omega^\infty$.

Note that every point $x \in \Omega^\infty$ by construction belongs to infinitely many of the sets $\tilde{\Omega}_m$. Choose such a point x. Relabeling, we may assume $x \in \bigcap_{m\in\mathbb{N}} \tilde{\Omega}_m$.

By uniform boundedness $|\nabla u_m(x)| \le C$ there exists a subsequence $m \to \infty$ and a vector $p \in \mathbb{R}^{nN}$ such that $\nabla u_m(x) \to p \ \ (m \to \infty)$. But then by continuity

$$F\big(x, u_m(x), \nabla u_m(x)\big) \to F\big(x, u(x), p\big)$$

while also

$$F\big(x, u(x), \nabla u_m(x)\big) \to F\big(x, u(x), p\big)$$

which contradicts the characterization of Ω_m given above. □

1.8 Remarks. The following observations may be useful in applications.
(1°) Theorem 1.6 also applies to functionals involving higher (mth-) order derivatives of a function u by letting $U = (u, \nabla u, \dots, \nabla^{m-1}u)$ denote the $(m-1)$-jet of u. Note that convexity is only required in the highest-order derivatives $P = \nabla^m u$.
(2°) If (u_m) is bounded in $H^{1,1}(\Omega')$ for any $\Omega' \subset\subset \Omega$, by Rellich's theorem and repeated selection of subsequences there exists a subsequence (u_m) which converges strongly in $L^1(\Omega')$ for any $\Omega' \subset\subset \Omega$.

Local boundedness in $H^{1,1}$ of a minimizing sequence (u_m) for E can be inferred from a coerciveness condition like

$$F(x, z, p) \ge |p|^\mu - \phi(x), \ \mu \ge 1, \ \phi \in L^1 \ . \tag{1.7}$$

The delicate part in the hypotheses concerning (u_m) is the assumption that (∇u_m) converges weakly in L^1_{loc}. In case $\mu > 1$ in (1.7) this is clear, but in case $\mu = 1$ the local L^1-limit of a minimizing sequence may lie in BV_{loc} instead of $H^{1,1}_{loc}$. See Theorem 1.4, for example; see also Section 3.
(3°) By convexity in p, continuity of F in (u, p) for almost every x is equivalent to the following condition, which is easier to check in applications:

> $F(x, \cdot, \cdot)$ is continuous, separately in $u \in \mathbb{R}^N$ and $p \in \mathbb{R}^{nN}$, for almost every $x \in \Omega$.

Indeed, for any fixed x, u, p and all $e \in \mathbb{R}^{nN}, |e| = 1, \ \alpha \in [0,1]$, letting $q = p + \alpha e, \ p_+ = p + e, \ p_- = p - e$ and writing $F(x, u, p) = F(u, p)$ for brevity, by convexity we have

$$F(u,q) = F(u, \alpha p_+ + (1-\alpha)p) \le \alpha F(u, p_+) + (1-\alpha)F(u,p) \ ,$$
$$F(u,p) = F(u, \frac{1}{1+\alpha}q + \frac{\alpha}{1+\alpha}p_-) \le \frac{1}{1+\alpha}F(u,q) + \frac{\alpha}{1+\alpha}F(u,p_-) \ .$$

Hence

$$\alpha\,(F(u,p) - F(u,p_+)) \le F(u,p) - F(u,q) \le \alpha\,(F(u,p_-) - F(u,p))$$

and it follows that

$$\sup_{|q-p|\le 1} \frac{|F(u,q) - F(u,p)|}{|q-p|} \le \sup_{|q-p|=1} |F(u,q) - F(u,p)| \ .$$

Since the sphere of radius 1 around p lies in the convex hull of finitely many vectors $q_0, q_1, \ldots, q_{nN}$, by continuity of F in u and convexity in p the right-hand side of this inequality remains uniformly bounded in a neighborhood of (u,p). Hence $F(\cdot,\cdot)$ is locally Lipschitz continous in p, locally uniformly in $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Therefore, if $u_m \to u$, $p_m \to p$ we have

$$\begin{aligned} |F(u_m,p_m) - F(u,p)| &\le |F(u_m,p_m) - F(u_m,p)| + |F(u_m,p) - F(u,p)| \\ &\le c|p_m - p| + o(1) \to 0 \qquad \text{as } m \to \infty, \end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$, as desired.
(4°) In the scalar case ($N = 1$), if F is C^2 for example, the existence of a minimizer u for E implies that the *Legendre condition*

$$\sum_{\alpha,\beta=1}^{n} F_{p_\alpha p_\beta}(x,u,p)\,\xi_\alpha \xi_\beta \ge 0, \qquad \text{for all } \xi \in \mathbb{R}^n$$

holds at all points $(x, u = u(x), p = \nabla u(x))$, see for instance Giaquinta [1; p. 11 f.]. This condition in turn implies the convexity of F in p.

The situation is quite different in the vector-valued case $N > 1$. In this case, in general only the *Legendre-Hadamard condition*

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1} F_{p^i_\alpha p^j_\beta}(x,u,p)\xi_\alpha \xi_\beta \eta^i \eta^j \ge 0\ , \qquad \text{for all } \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^N$$

will hold at a minimizer, which is much weaker then convexity (Giaquinta [1; p. 12]).

In fact, in Section 3 below we shall see how, under certain additional structure conditions on F, the convexity assumption in Theorem 1.6 can be weakened in the vector-valued case.

2. Constraints

Applying the direct methods often involves a delicate interplay between the functional E, the space of admissible functions M, and the topology on M. In this section we will see how, by means of imposing constraints on admissible functions and/or by a suitable modification of the variational problem, the direct methods can be successfully employed also in situations where their use seems highly unlikely at first.

Note that we will not consider constraints that are dictated by the problems themselves, such as physical restrictions on the response of a mechanical system. Constraints of this type in general lead to variational inequalities, and we refer to Kinderlehrer-Stampacchia [1] for a comprehensive introduction to this field. Instead, we will show how certain variational problems can be solved

by adding virtual – that is, purely technical – constraints to the conditions defining the admissible set, thus singling out distinguished solutions.

Semilinear Elliptic Boundary Value Problems

We start by deriving the existence of positive solutions to non-coercive, semilinear elliptic boundary value problems by a constrained minimization method. Such problems are motivated by studies of flame propagation (see for example Gel'fand [1; (15.5), p. 357]) or arise in the context of the Yamabe problem (see Section III.4).

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and let $p > 2$. If $n \geq 3$ we also assume that p satisfies the condition $p < 2^* = \frac{2n}{n-2}$. For $\lambda \in \mathbb{R}$ consider the problem

$$-\Delta u + \lambda u = u|u|^{p-2} \qquad \text{in } \Omega \,, \tag{2.1}$$

$$u > 0 \qquad \text{in } \Omega \,, \tag{2.2}$$

$$u = 0 \qquad \text{on } \partial\Omega \,. \tag{2.3}$$

Also let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ denote the eigenvalues of the operator $-\Delta$ on $H_0^{1,2}(\Omega)$. Then we have the following result:

2.1 Theorem. *For any $\lambda > -\lambda_1$ there exists a positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to problem (2.1)–(2.3).*

Proof. Observe that Equation (2.1) is the Euler-Lagrange equation of the functional

$$\tilde{E}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda|u|^2\right) dx - \frac{1}{p}\int_\Omega |u|^p \, dx$$

on $H_0^{1,2}(\Omega)$, which is neither bounded from above nor from below on this space. However, using the homogeneity of (2.1) a solution of problem (2.1)–(2.3) can also be obtained by solving a constrained minimization problem for the functional

$$E(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda|u|^2\right) dx$$

on the Hilbert space $H_0^{1,2}(\Omega)$, restricted to the set

$$M = \{u \in H_0^{1,2}(\Omega) \,;\, \int_\Omega |u|^p \, dx = 1\} \,.$$

We verify that $E : M \to \mathbb{R}$ satisfies the hypotheses of Theorem 1.2. By the Rellich-Kondrakov theorem the injection $H_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ is completely continuous for $p < 2^*$, if $n \geq 3$, respectively for any $p < \infty$, if $n = 1, 2$; see Theorem A.5 of Appendix A. Hence M is weakly closed in $H_0^{1,2}(\Omega)$.

Recall the Rayleigh-Ritz characterization

$$\lambda_1 = \inf_{\substack{u \in H_0^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} \tag{2.4}$$

of the smallest Dirichlet eigenvalue. This gives the estimate

$$E(u) \geq \frac{1}{2} \min\{1, \big(1 + \frac{\lambda}{\lambda_1}\big)\} \|u\|^2_{H_0^{1,2}} \ . \tag{2.5}$$

From this, coerciveness of E for $\lambda > -\lambda_1$ is immediate.

Weak lower semi-continuity of E follows from weak lower semi-continuity of the norm in $H_0^{1,2}(\Omega)$ and the Rellich-Kondrakov theorem. By Theorem 1.2 therefore E attains its infimum at a point $\underline{u}$ in M. Note that since $E(u) = E(|u|)$ we may assume that $\underline{u} \geq 0$.

To derive the variational equation for E first note that E is continuously Fréchet-differentiable in $H_0^{1,2}(\Omega)$ with

$$\langle v, DE(u) \rangle = \int_\Omega \big(\nabla u \nabla v + \lambda u v \big) \, dx \ .$$

Moreover, letting

$$G(u) = \int_\Omega |u|^p \, dx - 1 \ ,$$

for $p \leq 2^*$ also $G : H_0^{1,2}(\Omega) \to \mathbb{R}$ is continuously Fréchet-differentiable with

$$\langle v, DG(u) \rangle = p \int_\Omega u |u|^{p-2} v \, dx \ .$$

In particular, at any point $u \in M$

$$\langle u, DG(u) \rangle = p \int_\Omega |u|^p \, dx = p \neq 0 \ ,$$

and by the implicit function theorem the set $M = G^{-1}(0)$ is a C^1-submanifold of $H_0^{1,2}(\Omega)$.

Now, by the Lagrange multiplier rule, there exists a parameter $\mu \in \mathbb{R}$ such that

$$\begin{aligned} \langle v, (DE(\underline{u}) - \mu DG(\underline{u})) \rangle &= \int_\Omega \big(\nabla \underline{u} \nabla v + \lambda \underline{u} v - \mu \underline{u} |\underline{u}|^{p-2} v \big) \, dx \\ &= 0, \qquad \text{for all } v \in H_0^{1,2}(\Omega) \ . \end{aligned}$$

Inserting $v = \underline{u}$ into this equation yields that

$$2E(\underline{u}) = \int_\Omega \big(|\nabla \underline{u}|^2 + \lambda |\underline{u}|^2 \big) \, dx = \mu \int_\Omega |\underline{u}|^p \, dx = \mu \ .$$

Since $\underline{u} \in M$ cannot vanish identically, from (2.5) we infer that $\mu > 0$. Scaling with a suitable power of μ, we obtain a weak solution $u = \mu^{\frac{1}{p-2}} \cdot \underline{u} \in H_0^{1,2}(\Omega)$ of (2.1), (2.3) in the sense that

$$\int_\Omega \left(\nabla u \nabla v + \lambda uv - u|u|^{p-2}v\right) dx = 0 \ , \qquad \text{for all } v \in H_0^{1,2}(\Omega) \ . \tag{2.6}$$

Moreover, (2.2) holds in the weak sense $u \geq 0,\ u \neq 0$. To finish the proof we use the regularity result Lemma B.3 of Appendix B and the observations following it to obtain that $u \in C^2(\Omega)$. Finally, by the strong maximum principle $u > 0$ in Ω ; see Theorem B.4. □

Observe that, at least for the kind of nonlinear problems considered here, by Lemma B.3 of Appendix B the regularity theory is taken care of and in the following we may concentrate on proving existence of (weak) solutions. However, additional structure conditions may imply further useful properties of suitable solutions. An example is symmetry.

2.2 Symmetry. By a result of Gidas-Ni-Nirenberg [1; Theorem 2.1, p. 216, and Theorem 1, p. 209], if Ω is convex and symmetric with respect to a hyperplane, say $x_1 = 0$, any positive solution u of (2.1), (2.3) is even in x_1, that is, $u(x_1, x') = u(-x_1, x')$ for all $x = (x_1, x') \in \Omega$, and $\frac{\partial u}{\partial x_1} < 0$ at any point $x = (x_1, x') \in \Omega$ with $x_1 > 0$. In particular, if Ω is a ball, any positive solution u is radially symmetric. The proof of this result uses a variant of the Alexandrov-Hopf reflection principle and the maximum principle. This method lends itself to numerous applications in many different contexts; in Section III.4 we shall see that it is even possible to derive a-priori bounds from this method in the setting of a parabolic equation on the sphere.

Perron's Method in a Variational Guise

In the previous example the constraint built into the definition of M had the effect of making the restricted functional $E = \tilde{E}|_M$ coercive. Moreover, this constraint only changed the Euler-Lagrange equations by a factor which could be scaled away using the homogeneity of the right-hand side of (2.1).

In the next application we will see that sometimes also inequality constraints can be imposed without changing the Euler-Lagrange equations at a minimizer.

2.3 Weak sub- and super-solutions. Suppose Ω is a smooth, bounded domain in $\mathbb{R}^n$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the property that $|g(x,u)| \leq C(R)$ for any $R > 0$ and all u such that $|u(x)| \leq R$ almost everywhere. Given $u_0 \in H_0^{1,2}(\Omega)$, we then consider the equation

$$-\Delta u = g(\cdot, u) \qquad \text{in } \Omega \ , \tag{2.7}$$

$$u = u_0 \qquad \text{on } \partial\Omega \ . \tag{2.8}$$

By definition $u \in H^{1,2}(\Omega)$ is a (weak) *sub-solution* to (2.7–2.8) if $u \le u_0$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \nabla\varphi \, dx - \int_\Omega g(\,\cdot\,, u)\varphi \, dx \le 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega)\ ,\ \varphi \ge 0\ .$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) *super-solution* to (2.7–2.8) if in the above the reverse inequalities hold.

2.4 Theorem. *Suppose $\underline{u} \in H^{1,2}(\Omega)$ is a sub-solution while $\overline{u} \in H^{1,2}(\Omega)$ is a super-solution to problem (2.7–2.8) and assume that with constants $\underline{c}, \overline{c} \in \mathbb{R}$ there holds $-\infty < \underline{c} \le \underline{u} \le \overline{u} \le \overline{c} < \infty$, almost everywhere in Ω. Then there exists a weak solution $u \in H^{1,2}(\Omega)$ of (2.7–2.8), satisfying the condition $\underline{u} \le u \le \overline{u}$ almost everywhere in Ω.*

Proof. With no loss of generality we may assume $u_0 = 0$. Let $G(x,u) = \int_0^u g(x,v)\,dv$ denote a primitive of g. Note that (2.7–2.8) formally are the Euler-Lagrange equations of the functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(x,u)\, dx\ .$$

However, our assumptions do not allow the conclusion that E is finite or even differentiable on $V := H_0^{1,2}(\Omega)$ – the smallest space where we have any chance of verifying coerciveness. Instead we restrict E to

$$M = \big\{ u \in H_0^{1,2}(\Omega)\ ;\ \underline{u} \le u \le \overline{u} \text{ almost everywhere} \big\}\ .$$

Since $\underline{u}, \overline{u} \in L^\infty$ by assumption, also $M \subset L^\infty$ and $G\big(x, u(x)\big) \le c$ for all $u \in M$ and almost every $x \in \Omega$.

Now we can verify the hypotheses of Theorem 1.2: Clearly, $V = H_0^{1,2}(\Omega)$ is reflexive. Moreover, M is closed and convex, hence weakly closed. Since M is essentially bounded, our functional $E(u) \ge \frac{1}{2}\|u\|^2_{H_0^{1,2}(\Omega)} - c$ is coercive on M. Finally, to see that E is weakly lower semi-continuous on M, it suffices to show that

$$\int_\Omega G(x,u_m)\, dx \to \int_\Omega G(x,u)\, dx$$

if $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega)$, where $u_m, u \in M$. But – passing to a subsequence, if necessary – we may assume that $u_m \to u$ pointwise almost everywhere; moreover, $|G\big(x, u_m(x)\big)| \le c$ uniformly. Hence we may appeal to Lebesgue's theorem on dominated convergence.

From Theorem 1.2 we infer the existence of a relative minimizer $u \in M$. To see that u weakly solves (2.7), for $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$ let $v_\varepsilon = \min\big\{\overline{u}, \max\{\underline{u}, u + \varepsilon\varphi\}\big\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in M$ with

$$\varphi^\varepsilon = \max\big\{0, u + \varepsilon\varphi - \overline{u}\big\} \ge 0\ ,$$
$$\varphi_\varepsilon = \max\big\{0, \underline{u} - (u + \varepsilon\varphi)\big\} \ge 0\ .$$

Note that $\varphi_\varepsilon, \varphi^\varepsilon \in H_0^{1,2} \cap L^\infty(\Omega)$.

E is differentiable in direction $v_\varepsilon - u$. Since u minimizes E in M we have

$$0 \le \langle (v_\varepsilon - u), DE(u)\rangle = \varepsilon\langle \varphi, DE(u)\rangle - \langle \varphi^\varepsilon, DE(u)\rangle + \langle \varphi_\varepsilon, DE(u)\rangle \ ,$$

so that

$$\langle \varphi, DE(u)\rangle \ge \frac{1}{\varepsilon}\big[\langle \varphi^\varepsilon, DE(u)\rangle - \langle \varphi_\varepsilon, DE(u)\rangle\big] \ .$$

Now, since $\overline{u}$ is a supersolution to (2.7), we have

$$\begin{aligned}
\langle \varphi^\varepsilon, DE(u)\rangle &= \langle \varphi^\varepsilon, DE(\overline{u})\rangle + \langle \varphi^\varepsilon, DE(u) - DE(\overline{u})\rangle \\
&\ge \langle \varphi^\varepsilon, DE(u) - DE(\overline{u})\rangle \\
&= \int_{\Omega_\varepsilon} \big\{\nabla(u-\overline{u})\nabla(u+\varepsilon\varphi-\overline{u}) \\
&\qquad - \big(g(x,u) - g(x,\overline{u})\big)(u+\varepsilon\varphi-\overline{u})\big\}\, dx \\
&\ge \varepsilon\int_{\Omega_\varepsilon} \nabla(u-\overline{u})\nabla\varphi\, dx - \varepsilon\int_{\Omega_\varepsilon} \big|g(x,u) - g(x,\overline{u})\big|\,|\varphi|\, dx \ ,
\end{aligned}$$

where $\Omega^\varepsilon = \big\{x \in \Omega \ ; \ u(x) + \varepsilon\varphi(x) \ge \overline{u}(x) > u(x)\big\}$. Note that $\mathcal{L}^n(\Omega^\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence by absolute continuity of the Lebesgue integral we obtain that

$$\langle \varphi^\varepsilon, DE(u)\rangle \ge o(\varepsilon) \ ,$$

where $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. Similarly, we conclude that

$$\langle \varphi_\varepsilon, DE(u)\rangle \le o(\varepsilon) \ ,$$

whence

$$\langle \varphi, DE(u)\rangle \ge 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Reversing the sign of φ and since $C_0^\infty(\Omega)$ is dense in $H_0^{1,2}(\Omega)$ we finally see that $DE(u) = 0$, as claimed. □

2.5 A special case. Let Ω be a smooth bounded domain in $\mathbb{R}^n, n \ge 3$, and let

$$g(x,u) = k(x)u - u|u|^{p-2} \ , \tag{2.9}$$

where $p = \frac{2n}{n-2}$, and where k is a continuous function such that

$$1 \le k(x) \le K < \infty$$

uniformly in Ω. Suppose $u_0 \in C^1(\overline{\Omega})$ satisfies $u_0 \ge 1$ on $\partial\Omega$.
Then $\underline{u} \equiv 1$ is a sub-solution while $\overline{u} \equiv c$ for large $c > 1$ is a super-solution to Equations (2.7)–(2.8). Consequently, (2.7)–(2.8) admits a solution $u \ge 1$.

2.6 Remark. The sub-super-solution method can also be applied to equations on manifolds. In the context of the Yamabe problem it has been used by Loewner-Nirenberg [1] and Kazdan-Warner [1]; see Section III.4. The nonlinear term in this case is precisely (2.9).

The Classical Plateau Problem

One of the great successes of the direct methods in the calculus of variations was the solution of Plateau's problem for minimal surfaces.

Let Γ be a smooth Jordan curve in $\mathbb{R}^3$. From his famous experiments with soap films Plateau became convinced that any such curve is spanned by a (not necessarily unique) surface of least area.

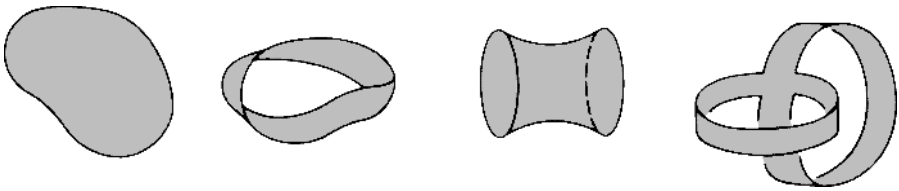

Fig. 2.1. Minimal surfaces of various topological types (disk, Möbius band, annulus, torus)

In the classical mathematical model the topological type of the surface is specified to be that of the disk

$$\Omega = \{z = (x, y) \; ; \; x^2 + y^2 < 1\} \; .$$

A naive approach to Plateau's conjecture would be to attempt to minimize the area

$$A(u) = \int_\Omega \sqrt{\det(\nabla u^t \nabla u)}\, dz = \int_\Omega \sqrt{|u_x|^2 |u_y|^2 - (u_x \cdot u_y)^2}\, dz$$

among "surfaces" $u \in H^{1,2} \cap C^0(\overline{\Omega}, \mathbb{R}^3)$ satisfying the Plateau boundary condition

$$u\big|_{\partial\Omega} : \partial\Omega \to \Gamma \qquad \text{is a (weakly) monotone parametrization preserving the given orientation of } \Gamma. \tag{2.10}$$

However, A is invariant under arbitrary changes of parameter. Hence there is no chance of achieving bounded compactness in the original variational problem and some work was necessary in order to recast this problem in a way which is accessible by direct methods. Without entering into details let us briefly report the main ideas.

It had already beeen observed by Lagrange that if a (smooth) surface S is (locally) area-minimizing for fixed boundary Γ, necessarily the mean curvature of S vanishes. In isothermal coordinates $u(x, y)$ on S this amounts to the equation

$$\Delta u = 0 \, . \tag{2.11}$$

(See Nitsche [2] or Osserman [1].) Moreover, our choice of parameter implies the conformality relations

$$|u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y \text{ in } \Omega\,, \tag{2.12}$$

in addition to the Plateau boundary condition (2.10).

We now take equations (2.10)–(2.12) as a definition for a minimal surface spanning Γ.

2.7 The variational problem. In their 1930 break-through papers, Douglas [1] and Radó [1] ingeniously proposed to solve (2.10)–(2.12) by minimizing Dirichlet's integral

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dz = \frac{1}{2}\int_\Omega (|u_x|^2 + |u_y|^2)\, dz$$

over the class

$$C(\Gamma) = \{u \in H^{1,2}(\Omega, \mathbb{R}^3)\ ;\ u\Big|_{\partial\Omega} \in C^0(\partial\Omega, \mathbb{R}^3) \text{ satisfies (2.10)}\}\ .$$

It is easy to see that

$$E(u) \geq A(u)\ ,$$

and equality holds if and only if u is conformal. Actually, we have

$$\inf_{C(\Gamma)} A(u) = \inf_{C(\Gamma)} E(u)\ .$$

This can be derived for instance from Morrey's "ε-conformality lemma" (Morrey [2; Theorem 1.2]). In Struwe [18; Appendix A] also a direct (variational) proof is given. Thus, a minimizer of E also will minimize A – hence it will satisfy (2.12) and solve the original minimization problem.

The solution of Plateau's problem is therefore reduced to the following theorem.

2.8 Theorem. *For any C^1-embedded curve Γ there exists a minimizer u of Dirichlet's integral E in $C(\Gamma)$.*

Note that $C(\Gamma) \neq \emptyset$ if $\Gamma \in C^1$. (Actually it suffices to assume that Γ is a rectifiable Jordan curve; see Douglas [1], Radó [1].)

To show Theorem 2.8, observe that in replacing A by E we have succeeded in reducing the symmetries of the problem drastically. However, E is still conformally invariant, that is

$$E(u) = E(u \circ g)$$

for all $g \in \mathcal{G}$, where

$$\mathcal{G} = \left\{ g : z \mapsto g(z) = e^{i\phi}\frac{a+z}{1-\overline{a}z}\ ;\ a \in \mathbb{C},\ |a| < 1,\ 0 \leq \phi < 2\pi \right\}$$

denotes the conformal group of Möbius transformations of the disc, viewed as a subset of $\mathbb{C}$. The action of $\mathcal{G}$ is non-compact in the sense that for any $u \in C(\Gamma)$ the orbit $\{u \circ g\,;\ g \in \mathcal{G}\}$ weakly accumulates also at constant functions; see for instance Struwe [17; Lemma I.4.1] for a detailed proof. Hence $C(\Gamma)$ cannot be weakly closed in $H^{1,2}(\Omega, \mathbb{R}^3)$ and Theorem 1.2 cannot yet be applied.

Fortunately, we can also get rid of conformal invariance of E. Note that for any oriented triple $e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3} \in \partial\Omega$, $0 \le \phi_1 < \phi_2 < \phi_3 < 2\pi$ there exists a unique $g \in \mathcal{G}$ such that $g\left(e^{\frac{2\pi ik}{3}}\right) = e^{i\phi_k}$, $k = 1, 2, 3$.

Fix a parametrization γ of Γ (we may assume that γ is a C^1-diffeomorphism $\gamma : \partial\Omega \to \Gamma$) and let

$$C^*(\Gamma) = \left\{u \in C(\Gamma)\ ;\ u\left(e^{\frac{2\pi ik}{3}}\right) = \gamma\left(e^{\frac{2\pi ik}{3}}\right),\ k = 1, 2, 3\right\}\ ,$$

endowed with the $H^{1,2}$-topology. This is our space of admissible functions, normalized with respect to $\mathcal{G}$. Note that for any $u \in C(\Gamma)$ there is $g \in \mathcal{G}$ such that $u \circ g \in C^*(\Gamma)$.

The following result now is a consequence of the classical "Courant-Lebesgue lemma".

2.9 Lemma. *The set $C^*(\Gamma)$ is weakly closed in $H^{1,2}$.*

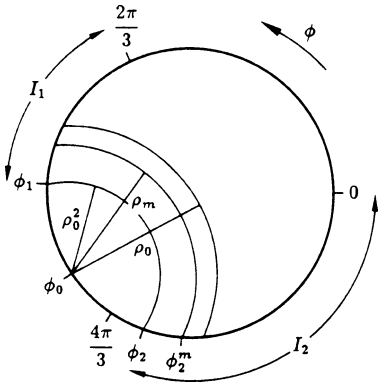

Fig. 2.2.

Proof. The proof in a subtle way uses a convexity argument as in the preceding example. To present this argument explicitly we use the fixed parametrization γ to associate with any $u \in C^*(\Gamma)$ a continuous map $\xi : \mathbb{R} \to \mathbb{R}$, such that

$$\gamma\left(e^{i\xi(\phi)}\right) = u\left(e^{i\phi}\right)\ ,\ \xi(0) = 0\ .$$

By (2.10) the functions ξ obtained in this manner are continuous, monotone and $\xi - id$ is 2π-periodic; moreover, $\xi\left(\frac{2\pi k}{3}\right) = \frac{2\pi k}{3}$, for all $k \in \mathbb{Z}$ by our three-point normalization.

Now let

$$M = \big\{\xi : \mathbb{R} \to \mathbb{R} \ ; \ \xi \text{ is continuous and monotone,}$$
$$\xi(\phi + 2\pi) = \xi(\phi) + 2\pi, \ \xi\left(\frac{2\pi k}{3}\right) = \frac{2\pi k}{3}, \text{ for all } \phi \in \mathbb{R}, k \in \mathbb{Z}\big\} \ .$$

Note that M is convex. Let (u_m) be a sequence in $C^*(\Gamma)$ with associated functions $\xi_m \in M$, and suppose $u_m \rightharpoonup u$ weakly in $H^{1,2}(\Omega)$. Since each ξ_m is monotone and satisfies the estimate $0 \le \xi_m(\phi) \le 2\pi$, for all $\phi \in [0, 2\pi]$, the family (ξ_m) is bounded in $BV\big([0, 2\pi]\big)$. Hence (a subsequence) $\xi_m \to \xi$ almost everywhere on $[0, 2\pi]$ and therefore – by periodicity – almost everywhere on $\mathbb{R}$, where ξ is monotone, $\xi - id$ is 2π-periodic, and ξ satisfies $\xi\left(\frac{2\pi k}{3}\right) = \frac{2\pi k}{3}$, for all $k \in \mathbb{Z}$.

Now if ξ is continuous, it follows from monotonicity that $\xi_m \to \xi$ uniformly. Thus, by continuity of γ, also u_m converges uniformly to u on $\partial\Omega$, and it follows that $u|_{\partial\Omega}$ is continuous and satisfies (2.10). That is, $u \in C^*(\Gamma)$, and the proof is complete in this case.

In order to exclude the remaining case, assume by contradiction that ξ is discontinuous at some point ϕ_0. We choose $k \in \mathbb{Z}$ such that $|\phi_0 - \frac{2\pi k}{3}| \le \frac{\pi}{3}$ and let $\big]\frac{2\pi(k-1)}{3}, \frac{2\pi(k+1)}{3}\big[=: I_0$. By monotonicity, for almost every $\phi_1, \phi_2 \in I_0$ such that $\phi_1 < \phi_0 < \phi_2$ we have

$$\begin{aligned}\frac{2\pi(k-1)}{3} &\le \lim_{m\to\infty} \xi_m(\phi_1) = \xi(\phi_1) \le \lim_{\phi\to\phi_0^-} \xi(\phi) \\ &< \lim_{\phi\to\phi_0^+} \xi(\phi) \le \xi(\phi_2) = \lim_{m\to\infty} \xi_m(\phi_2) \le \frac{2\pi(k+1)}{3} \ .\end{aligned}$$

For such $\phi_1, \ \phi_2 \in I_0$ denote $I_1 = \{\phi \in I_0 \ ; \ \phi \le \phi_1\}, \ \ I_2 = \{\phi \in I_0 \ ; \ \phi \ge \phi_2\}$. Then by monotonicity of ξ_m and using the fact that γ is a diffeomorphism we obtain

$$\begin{aligned}\limsup_{m\to\infty} &\left(\inf_{\phi\in I_1, \ \psi\in I_2} |\gamma\left(e^{i\xi_m(\phi)}\right) - \gamma\left(e^{i\xi_m(\psi)}\right)| \right) \\ &\ge \inf_{\phi,\psi\in I_0, \phi<\phi_0<\psi} |\gamma\left(e^{i\xi(\phi)}\right) - \gamma\left(e^{i\xi(\psi)}\right)| > 0 \ .\end{aligned}$$

In particular, there exists $\varepsilon > 0$ independent of ϕ_1, ϕ_2 such that

$$|u_m(e^{i\phi}) - u_m(e^{i\psi})| \ge \varepsilon > 0 \tag{2.13}$$

for all $\phi \in I_1, \ \psi \in I_2$ if $m \ge m_0(\phi_1, \phi_2)$ is sufficiently large.

Now let $z_0 = e^{i\phi_0}$ and for $\rho > 0$ denote

$$U_\rho = \{z \in \Omega \ ; \ |z - z_0| < \rho\}, \ C_\rho = \{z \in \overline{\Omega} \ ; \ |z - z_0| = \rho\} \ .$$

Note that for all $\rho < 1$ any point $z = e^{i\phi} \in C_\rho \cap \partial\Omega$ satisfies $\phi \in I_0$.

Following Courant [1; p. 103], we will use uniform boundedness of (u_m) in $H^{1,2}$ to show that for suitable numbers $\rho_0 \in]0, 1[, \ \rho_m \in [\rho_0^2, \rho_0]$ the oscillation of u_m on C_{ρ_m} can be made arbitrarily small, uniformly in $m \in \mathbb{N}$.

First note that by Fubini's theorem, if we denote arc length on C_ρ by s, from the estimate

$$\infty > c \geq \int_\Omega |\nabla u_m|^2 \, dz \geq \int_0^1 \int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds \, d\rho$$

we obtain that

$$\int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds < \infty$$

for almost every $\rho < 1$ and all $m \in \mathbb{N}$.
Choosing $\rho_0 < 1$ we may refine this estimate as follows:

$$\begin{aligned} \int_\Omega |\nabla u_m|^2 \, dz &\geq \int_{\rho_0^2}^{\rho_0} \left(\rho \int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds \right) \frac{d\rho}{\rho} \\ &\geq |\log \ \rho_0| \ \operatorname*{ess\,inf}_{\rho_0^2 \leq \rho \leq \rho_0} \left(\rho \int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds \right). \end{aligned}$$

Suppose $\rho_m \in [\rho_0^2, \rho_0]$ is such that

$$\rho_m \int_{C_{\rho_m}} |\frac{\partial}{\partial s} u_m|^2 \, ds \leq 2 \operatorname*{ess\,inf}_{\rho_0^2 \leq \rho \leq \rho_0} \left(\rho \int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds \right)$$

and denote

$$C = \sup_{m \in \mathbb{N}} \int_\Omega |\nabla u_m|^2 \, dz < \infty \ .$$

Fix $z_j = e^{i\phi_j}$, $j = 1, 2$, the points of intersection of $C_{\rho_0^2}$ with $\partial\Omega$, $\phi_1 < \phi_0 < \phi_2$. Also denote $z_j^m = e^{i\phi_j^m}$, $j = 1, 2$, with $\phi_1^m < \phi_0 < \phi_2^m$ the points of intersection of C_{ρ_m} with $\partial\Omega$.

Then $\phi_1^m \in I_1$, $\phi_2^m \in I_2$ while by Hölder's inequality

$$\begin{aligned} |u_m(z_1^m) - u_m(z_2^m)|^2 &\leq \left(\int_{C_{\rho_m}} |\frac{\partial}{\partial s} u_m| \, ds \right)^2 \\ &\leq \pi \rho_m \int_{C_{\rho_m}} |\frac{\partial}{\partial s} u_m|^2 \, ds \leq \frac{2\pi C}{|\log \ \rho_0|} < \varepsilon \end{aligned} \tag{2.14}$$

if $\rho_0 > 0$ is sufficiently small. This estimate being uniform in m, for large $m \geq m_0(\phi_1, \phi_2)$ we obtain a contradiction to (2.13) and the proof is complete. □

Remark. From (2.14), monotonicity (2.10), the assumption that Γ is a Jordan curve – that is, a homeomorphic image of the circle S^1 – and the three-point condition, also a direct proof of equi-continuity of the sub-level sets of E in $C^*(\Gamma)$ can be given; see Courant [1; Lemma 3.2, p. 103] or Struwe [17; Lemma I.4.3]. Moreover, note that (2.14) implies a uniform estimate for the modulus of continuity of a function $u \in H_0^{1,2}(\Omega)$ on a sequence of concentric circular arcs around any fixed center $z_0 \in \overline{\Omega}$ in terms of its Dirichlet integral. This observation was used by Lebesgue [1] to obtain an equi-continuous minimizing sequence for Dirichlet's integral in his solution of the classical Dirichlet problem; see also Section 5.7.

Proof of Theorem 2.8. By (2.10) and the generalized Poincaré inequality Theorem A.9 of Appendix A, for $u \in C^*(\Gamma)$ there holds

$$\int_\Omega |u|^2\, dz \le c\left(\int_\Omega |\nabla u|^2\, dz + \int_{\partial\Omega} |u|^2\, do\right) \le cE(u) + c(\Gamma).$$

Thus E is coercive on $M = C^*(\Gamma)$ in $H^{1,2}(\Omega;\mathbb{R}^3)$. Moreover E is weakly lower semi-continuous on $H^{1,2}(\Omega;\mathbb{R}^3)$. By Theorem 1.2 and Lemma 2.9 therefore the functional E achieves its infimum in $C^*(\Gamma)$, which by conformal invariance equals that in $C(\Gamma)$. □

2.10 Regularity. As in the preceding examples we may ask what regularity properties the minimizer u and the parametrized surface possess. We note a few results:
(1°) $u\big|_{\partial\Omega}$ is strictly monotone (Douglas [1], Radó [1]).
(2°) If $\Gamma \in C^{m,\alpha},\ m \ge 1,\ 0 < \alpha < 1$, then also $u \in C^{m,\alpha}(\overline{\Omega},\mathbb{R}^3)$; this result is due to Hildebrandt [1] (for $m \ge 4$) with later improvements by Nitsche [1] .
While remarks (1°), (2°) apply to arbitrary solutions of the Plateau problem (2.10)–(2.12), minimality is crucial for the next observations concerning the geometric regularity of the parametrized solution surface.
(3°) A minimizer $u \in C^*(\Gamma)$ parametrizes an immersed minimal surface $S = u(\Omega) \subset \mathbb{R}^3$; see Osserman [2], Gulliver [1], Alt [1], Gulliver-Osserman-Royden [1]; if Γ is anlytic, S is immersed up to the boundary; see Gulliver-Lesley [1]. If Γ is extreme, that is $\Gamma \subset \partial K$ where $K \subset \mathbb{R}^3$ is convex, S is embedded; see Meeks-Yau [1]. Existence of embedded minimal surfaces bounded by extreme curves independently was obtained by Tomi-Tromba [1] and Almgren-Simon [1].

2.11 Note. In the history of the calculus of variations it seems that Plateau's problem has played a very prominent role. Important developments in the general stream of ideas often were prompted by insights gained from the study of minimal surfaces. As an example, consider the classical mountain pass lemma (see also Chapter II.1) which was used by Courant [1; Chapter VI.6–7] to establish the existence of unstable minimal surfaces, previously obtained by

Morse-Tompkins [1] and Shiffman [1] by a less direct, topological reasoning. However, since this material has been covered very extensively elsewhere (see Struwe [18]), here we will confine ourselves to the above remarks.

For an introduction to Plateau's problem and minimal surfaces, see for instance, Osserman [1], or consult the encyclopedic book by Nitsche [2].

A truely remarkable – popular and profound – book on the subject is available by Hildebrandt-Tromba [1].

3. Compensated Compactness

As noted in Remark 1.8, it is conceivable that in the vector-valued case lower semi-continuity results may hold true under a weaker convexity assumption than in Theorem 1.6, provided suitable structure conditions are satisfied by the functional in variation. Weakening the convexity hypothesis is necessary, for instance, in dealing with problems arising in 3-dimensional elasticity, where we encounter energy functionals $\int_\Omega W(\nabla u)\,dx$ with a stored energy function W depending on the determinant, the minors and the eigenvalues of the deformation gradient ∇u. Since infinite volume distortion for elastic materials will afford an infinite amount of energy, it is natural to suppose that $W \to \infty$ if either $\det(\nabla u) \to 0$ or $\det(\nabla u) \to \infty$; hence W cannot be convex in ∇u. However, there is a large class of materials that can be described by *polyconvex* stored energy functions, which are of the form

$$W(\nabla u) = f(\text{subdeterminants of } \nabla u),$$

where f is convex in each of its variables. John Ball [1] was the first to see that lower semi-continuity results will hold for such functionals. The difficulty, of course, lies in proving, for instance, weak convergence $\det(\nabla u_m) \rightharpoonup \det(\nabla u)$ for a sequence $u_m \rightharpoonup u$ weakly in $H^{1,3}(\Omega, \mathbb{R}^3)$. Questions of this type had been investigated by Reshetnyak [1], [2]. A general frame for studying such problems is provided by the compensated compactness scheme of Murat and Tartar.

The basic principle of the compensated compactness method is given in the following lemma; see Tartar [2; p. 270 f.].

3.1 The compensated compactness lemma. *Let Ω be a domain in $\mathbb{R}^n$ and suppose that*
(1°) $u_m = \left(u_m^1, \dots, u_m^N\right) \rightharpoonup u$ weakly in $L^2(\Omega; \mathbb{R}^N)$.
(2°) The set $\left\{\sum_{j,k} a_{jk} \frac{\partial u_m^j}{\partial x_k}\ ;\ m \in \mathbb{N}\right\}$ is relatively compact in $H^{-1}_{loc}(\Omega; \mathbb{R}^L)$ for a set of vectors $a_{jk} \in \mathbb{R}^L;\ 1 \le j \le N,\ 1 \le k \le n$. Let

$$\Lambda = \left\{\lambda \in \mathbb{R}^N\ ;\ \sum_{j,k} a_{jk}\lambda_j\xi_k = 0 \text{ for some } \xi \in \mathbb{R}^n \setminus \{0\}\right\}$$

and let Q be a (real) quadratic form such that $Q(\lambda) \geq 0$ for all $\lambda \in \Lambda$. Regarding $Q(u_m) \in L^1(\Omega)$ as Radon measures $Q(u_m)dx \in \big(C^0(\Omega')\big)^*$, we may assume that $\big(Q(u_m)\big)$ converges weak, locally.*
Then on any $\Omega' \subset\subset \Omega$ we have

$$\operatorname*{weak}^*\!-\lim_{m\to\infty} Q\big(u_m\big) \geq Q(u)$$

in the sense of measures. In particular, if $Q(\lambda) = 0$ for all $\lambda \in \Lambda$, then

$$\operatorname*{weak}^*\!-\lim_{m\to\infty} Q\big(u_m\big) = Q(u)$$

locally, in the sense of measures.

Proof. Choose $\varphi \in C^0(\Omega)$ with compact support and such that $0 \leq \varphi \leq 1$. We must show that

$$\liminf_{m\to\infty} \int_\Omega Q\big(u_m\big)\varphi^2 \, dx \geq \int_\Omega Q(u)\varphi^2 \, dx \ .$$

By translation we may assume that $u = 0$. Moreover, upon replacing u_m by $u_m\varphi$ we may assume that the supports of u_m lie in a fixed cube $K \subset\subset \Omega$ and that $\varphi \equiv 1$ on K. By translation and scaling, moreover, we can obtain $K = [0, 2\pi]^n$. Let

$$u_m(x) = \sum_{\alpha\in\mathbb{Z}^n} \mu_{m,\alpha} e^{i\alpha\cdot x} \ , \qquad \mu_{m,\alpha} = \big(\mu^1_{m,\alpha}, \ldots, \mu^N_{m,\alpha}\big) \in \mathbb{C}^N \ ,$$

be the Fourier expansion for u_m. Since u_m is real, we have $\mu_{m,\alpha} = \overline{\mu}_{m,-\alpha}$. The assertion then is equivalent to showing that

$$\liminf_{m\to\infty} \frac{1}{(2\pi)^n} \int_K Q\big(u_m\big) \, dx = \liminf_{m\to\infty} \sum_{\alpha\in\mathbb{Z}^n} \Big(Q\big(\operatorname{Re} \mu_{m,\alpha}\big) + Q\big(\operatorname{Im} \mu_{m,\alpha}\big) \Big) \geq 0 \ .$$

By weak convergence $u_m \rightharpoonup 0$ in L^2 we infer that $\sum_{\alpha\in\mathbb{Z}^n} |\mu_{m,\alpha}|^2 \leq c < \infty$ and $\mu_{m,\alpha} \to 0$ as $m \to \infty$, uniformly on bounded sets of indices α.

Moreover, by (2°) the set

$$\left\{ \sum_{\alpha\in\mathbb{Z}^n} \sum_{j,k} a_{jk} \mu^j_{m,\alpha} \, \alpha_k \, e^{i\,\alpha\cdot x} \ ; \ m \in \mathbb{N} \right\}$$

is relatively compact in H^{-1}, which implies that

$$\sum_{\substack{\alpha\in\mathbb{Z}^n \\ |\alpha|\geq\alpha_0}} \frac{\Big|\sum_{j,k} a_{jk} \mu^j_{m,\alpha} \, \alpha_k\Big|^2}{1 + |\alpha|^2} \to 0,$$

as $\alpha_0 \to \infty$, uniformly in $m \in \mathbb{N}$. But this means that $\mu_{m,\alpha}$ can be decomposed $\mu_{m,\alpha} = \lambda_{m,\alpha} + \nu_{m,\alpha}$ with Re $\lambda_{m,\alpha}$, Im $\lambda_{m,\alpha} \in \Lambda$ and $\sum_{|\alpha|\geq\alpha_0} \left|\nu_{m,\alpha}\right|^2 \to 0$ as $\alpha_o \to \infty$, uniformly in $m \in \mathbb{N}$.

Indeed, for any $\alpha \in \mathbb{Z}^n$, $m \in \mathbb{N}$ decompose $\mu_{m,\alpha} = \lambda_{m,\alpha} + \nu_{m,\alpha}$, where Re $\lambda_{m,\alpha}$, Im $\lambda_{m,\alpha} \in \Lambda$ and $|\nu_{m,\alpha}|^2$ is minimal among all decompositions of this kind. We claim that for any $\varepsilon > 0$ there exist constants $C = C(\varepsilon)$, $\alpha_0 = \alpha_0(\varepsilon)$ such that for $|\alpha| \geq \alpha_0$ we can bound

$$|\nu_{m,\alpha}|^2 \leq C\left|\sum_{j,k} a_{jk}\mu^j_{m,\alpha}\frac{\alpha_k}{\sqrt{1+|\alpha|^2}}\right|^2 + \varepsilon\left|\mu_{m,\alpha}\right|^2 \tag{3.1}$$

uniformly in $m \in \mathbb{N}$.

Otherwise there exists $\varepsilon > 0$ and a sequence $\alpha = \alpha(l)$, $l \in \mathbb{N}$, with $|\alpha(l)| \geq l$, and $m = m(l)$ such that for all l there holds

$$|\mu_{m,\alpha}|^2 \geq |\nu_{m,\alpha}|^2 \geq l\left|\sum_{j,k} a_{jk}\mu^j_{m,\alpha}\frac{\alpha_k}{\sqrt{1+|\alpha|^2}}\right|^2 + \varepsilon\,|\mu_{m,\alpha}|^2 \ . \tag{3.2}$$

(The first inequality follows from the choice of $\nu_{m,\alpha}$ above.) Let $\xi(l)$, $\eta(l)$ be the unit vectors

$$\xi(l) = \frac{\alpha(l)}{\sqrt{1+|\alpha(l)|^2}} \in S^{n-1},\ \eta(l) = \frac{\mu_{m(l),\alpha(l)}}{\left|\mu_{m(l),\alpha(l)}\right|} \in S^{N-1}\ ,$$

and denote by $A(l)\colon \mathbb{R}^N \to \mathbb{R}^L$ the linear map

$$\eta \mapsto \sum_{j,k} a_{jk}\eta_j\xi_k(l)\ .$$

We may assume that $\xi(l) \to \xi$ and $A(l) \to A$ as $l \to \infty$. Likewise, we may suppose that $\eta(l) \to \eta$. Passing to the limit in (3.2) it follows that $\eta \in ker\ A$; that is, $\eta \in \Lambda$. Projecting $\mu_{m,\alpha}$ onto $ker\ A$ for all $\alpha = \alpha(l)$, $m = m(l)$ we hence obtain a decomposition $\mu_{m,\alpha} = \tilde{\lambda}_{m,\alpha} + \tilde{\nu}_{m,\alpha}$ with Re $\tilde{\lambda}_{m,\alpha}$, Im $\tilde{\lambda}_{m,\alpha} \in ker\ A \subset \Lambda$ and

$$\begin{aligned}\left|\tilde{\nu}_{m,\alpha}\right|^2 &\leq C\left|A\mu_{m,\alpha}\right|^2 \leq C\left|A(l)\mu_{m,\alpha}\right|^2 + o(1)\left|\mu_{m,\alpha}\right|^2 \\ &= C\left|\sum_{j,k} a_{jk}\mu^j_{m,\alpha}\frac{\alpha_k}{\sqrt{1+|\alpha|^2}}\right|^2 + o(1)\left|\mu_{m,\alpha}\right|^2\ ,\end{aligned}$$

where $o(1) \to 0$ $(l \to \infty)$. But by defintion $\left|\nu_{m,\alpha}\right|^2 \leq \left|\tilde{\nu}_{m,\alpha}\right|^2$, and we obtain the desired contradiction to assumption (3.2). Hence for any $\varepsilon > 0$, any $\alpha_0 \geq \alpha_0(\varepsilon)$:

$$\begin{aligned}
\liminf_{m\to\infty} &\sum_{\alpha\in\mathbb{Z}^n}\Big(Q\big(\mathrm{Re}\;\mu_{m,\alpha}\big)+Q\big(\mathrm{Im}\;\mu_{m,\alpha}\big)\Big)\\
&=\liminf_{m\to\infty}\sum_{|\alpha|\geq\alpha_0}\Big(Q\big(\mathrm{Re}\;\mu_{m,\alpha}\big)+Q\big(\mathrm{Im}\;\mu_{m,\alpha}\big)\Big)\\
&\geq\liminf_{m\to\infty}\sum_{|\alpha|\geq\alpha_0}\Big(Q\big(\mathrm{Re}\;\lambda_{m,\alpha}\big)+Q\big(\mathrm{Im}\;\lambda_{m,\alpha}\big)-c\varepsilon\big|\mu_{m,\alpha}\big|^2\Big)+o(1)\\
&\geq o(1)-c\varepsilon\;,
\end{aligned}$$

where $o(1)\to 0$ as $\alpha_0\to\infty$. Thus, the assertion of the lemma follows as we first let $\alpha_0\to\infty$ and then $\varepsilon\to 0$. □

As an application we mention the following well-known result.

3.2 The Div-Curl Lemma. *Suppose $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ weakly in $L^2(\Omega;\mathbb{R}^3)$ on a domain $\Omega\subset\mathbb{R}^3$ while the sequences* (div u_m) *and* (curl v_m) *are relatively compact in $H^{-1}(\Omega)$. Then for any $\varphi\in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u_m\cdot v_m\varphi\,dx\to\int_\Omega u\cdot v\varphi\,dx$$

as $m\to\infty$.

Proof. Let $w_m=(u_m,v_m)\in L^2(\Omega;\mathbb{R}^6)$, and determine coefficients $a_{jk}\in\mathbb{R}^4$ such that $\sum_{jk}a_{jk}\frac{\partial w_m^j}{\partial x_k}=(\mathrm{div}\;u_m,\;\mathrm{curl}\;v_m)$. Let Q be the quadratic form $Q(u,v)=u\cdot v$, acting on vectors $w=(u,v)\in\mathbb{R}^6$. Note $a_{jk}=\big(\delta_{jk},(\varepsilon_{ijk})_{1\leq i\leq 3}\big)$ where $\delta_{jk}=1$ if $j=k$, and $\delta_{jk}=0$ else, $\varepsilon_{123}=1$ and $\varepsilon_{ijk}=-\varepsilon_{jik}=\varepsilon_{jki}$. Hence

$$\begin{aligned}
\Lambda&=\big\{\lambda=(\mu,\nu)\in\mathbb{R}^6\;;\;\exists\xi\in\mathbb{R}^3\setminus\{0\}:\;(\xi\cdot\mu\;,\;\xi\wedge\nu)=0\big\}\\
&=\big\{\lambda=(\mu,\nu)\in\mathbb{R}^6\;;\;\mu\cdot\nu=0\big\}\;,
\end{aligned}$$

and $Q\equiv 0$ on Λ. Thus the assertion follows from Lemma 3.1. □

The div-curl Lemma 3.2 shows how additional bounds on *some* derivatives allow one to prove continuity of nonlinear expressions (bi-linear in the above example) under weak convergence.

Phrased somewhat differently, the reason for the convergence result stated in the div-curl lemma to hold is an implicit divergence structure. This stucture can be brought out more clearly in the language of differential forms. For simplicity, we assume that all functions involved are periodic of period 1 in each variable. In this case, we may regard $\Omega=[0,1]^n$ as a fundamental domain for the flat n-dimensional torus $T^n=\mathbb{R}^n/\mathbb{Z}^n$. Let d, d^* be the exterior differential and co-differential, respectively. We consider 1-forms $u_m\rightharpoonup u$ in L^2, $v_m\rightharpoonup v$

in L^2 weakly as $m \to \infty$ with $(d^*u_m), (dv_m)$ relatively compact in H^{-1}. We may assume $u = 0, v = 0$. By Hodge decomposition we have

$$u_m = da_m + d^*b_m + c_m,$$
$$v_m = df_m + d^*g_m + h_m,$$

where c_m, h_m are harmonic 1-forms, $db_m = dg_m = 0$, and with $a_m \rightharpoonup 0, b_m \rightharpoonup 0, f_m \rightharpoonup 0, g_m \rightharpoonup 0$ weakly in $H^{1,2}(T^n)$, $c_m \rightharpoonup 0$, $h_m \rightharpoonup 0$ weakly in $L^2(T^n)$. Since the space of harmonic 1-forms on T^n is compact, in fact, $c_m \to 0$ and $h_m \to 0$ smoothly as $m \to \infty$. Moreover, since the sequences

$$\Delta a_m = d^*u_m, \Delta g_m = dv_m$$

are relatively compact in H^{-1}, it follows that $(da_m), (d^*g_m)$ are precompact in L^2, and we conclude that

$$u_m = d^*b_m + o(1), v_m = df_m + o(1),$$

where $o(1) \to 0$ in L^2 as $m \to \infty$. Moreover, using the Hodge $*$-operator and denoting by "$\cdot$" contraction of forms, we have

$$u_m \cdot v_m \, dx + o(1) = *(d^*b_m \cdot df_m) = (d * b_m) \wedge df_m = d\big((*b_m) \wedge df_m\big) \ ,$$

thus exhibiting the asserted divergence structure. Since by Rellich's compactness result $b_m \to 0$ strongly in L^2 as $m \to \infty$, it is now trivial to pass to the limit in the expression

$$\begin{aligned}\int_{T^n} u_m \cdot v_m \varphi \, dx &= \int_{T^n} d\big((*b_m \wedge df_m\big)\varphi + o(1) \\ &= (-1)^n \int_{T^n} (*b_m) \wedge df_m \wedge d\varphi + o(1) = o(1),\end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$.

A divergence structure is also the crucial ingredient in the applications that follow.

Applications in Elasticity

The most important applications of the compensated compactness method so far are in elasticity and hyperbolic systems, see Ball [1], [2], DiPerna [1]. DiPerna-Majda have applied compensated compactness methods to obtain the existence of weak solutions to the Euler equations for incompressible fluids, see for instance DiPerna-Majda [1]. Our interest lies with the extensions of the direct methods that compensated compactness implies. Thus we will concentrate on Ball's lower semi-continuity results for polyconvex materials in elasticity.

3.3 Theorem. *Suppose W is a function on (3×3)-matrices Φ, given by*

$$W(\Phi) = g(\Phi, adj\ \Phi, \det\ \Phi)$$

where g is a convex non-negative function in the sub-determinants of Φ. Let Ω be a domain in $\mathbb{R}^3$ and let $u_m,\ u \in H^{1,3}_{loc}(\Omega;\mathbb{R}^3)$. Suppose that $u_m \rightharpoonup u$ weakly in $H^{1,3}(\Omega';\mathbb{R}^3)$ while $\det(\nabla u_m) \rightharpoonup h$ weakly in $L^1(\Omega')$ for all $\Omega' \subset\subset \Omega$, where $h \in L^1_{loc}(\Omega)$. Then

$$\int_\Omega W(\nabla u)\,dx \le \liminf_{m\to\infty} \int_\Omega W(\nabla u_m)\,dx\ .$$

Proof. The proof of Theorem 1.6 can be carried over once we show that under the hypotheses made

$$adj(\nabla u_m) \rightharpoonup adj(\nabla u)$$
$$\det(\nabla u_m) \rightharpoonup \det(\nabla u)$$

weakly in $L^1(\Omega')$ for all $\Omega' \subset\subset \Omega$. The first assertion is a consequence of the divergence structure of the adjoint matrix $A_m = adj\,(\nabla u_m)$. Indeed, if indices i, j are counted *modulo* 3 we have

$$\begin{aligned} A^{ij}_m &= \frac{\partial u^{i+1}_m}{\partial x^{j+1}}\,\frac{\partial u^{i+2}_m}{\partial x^{j+2}} \;-\; \frac{\partial u^{i+2}_m}{\partial x^{j+1}}\,\frac{\partial u^{i+1}_m}{\partial x^{j+2}} \\ &= \frac{\partial}{\partial x^{j+1}}\left(u^{i+1}_m \frac{\partial u^{i+2}_m}{\partial x^{j+2}}\right) \;-\; \frac{\partial}{\partial x^{j+2}}\left(u^{i+1}_m \frac{\partial u^{i+2}_m}{\partial x^{j+1}}\right)\ . \end{aligned}$$

Fix $\Omega' \subset\subset \Omega$. Note that $\left(A^{ij}_m\right)$ is bounded in $L^{3/2}(\Omega')$. Hence we may assume that $A^{ij}_m \rightharpoonup A^{ij}$ weakly in $L^{3/2}(\Omega')$. Moreover, by Rellich's theorem $u_m \to u$ in $L^3(\Omega')$, whence $u_m \nabla u_m \rightharpoonup u\nabla u$ weakly in $L^{3/2}(\Omega')$. By continuity of the distributional derivative with respect to weak convergence therefore $A^{ij}_m \rightharpoonup \big(adj(\nabla u)\big)^{ij}$ in the sense of distributions. Finally, by uniqueness of the distributional limit, $A^{ij} = \big(adj(\nabla u)\big)^{ij}$, and $adj(\nabla u_m) \rightharpoonup adj(\nabla u)$ weakly in $L^{3/2}(\Omega')$, in particular weakly in $L^1(\Omega')$, as claimed.

Similarly, expanding the determinant along the first row, we have

$$\begin{aligned} \det(\nabla u_m) = \frac{\partial u^1_m}{\partial x^1}&\left[\frac{\partial u^2_m}{\partial x^2}\frac{\partial u^3_m}{\partial x^3} \;-\; \frac{\partial u^2_m}{\partial x^3}\frac{\partial u^3_m}{\partial x^2}\right] \\ -\frac{\partial u^1_m}{\partial x^2}&\left[\frac{\partial u^2_m}{\partial x^1}\frac{\partial u^3_m}{\partial x^3} \;-\; \frac{\partial u^2_m}{\partial x^3}\frac{\partial u^3_m}{\partial x^1}\right] \\ +\frac{\partial u^1_m}{\partial x^3}&\left[\frac{\partial u^2_m}{\partial x^1}\frac{\partial u^3_m}{\partial x^2} \;-\; \frac{\partial u^2_m}{\partial x^2}\frac{\partial u^3_m}{\partial x^1}\right]. \end{aligned}$$

Again, a divergence structure emerges, if we rewrite this as

$$\begin{aligned}\det(\nabla u_m) = &\frac{\partial}{\partial x^1}\left(u_m^1\left[\frac{\partial u_m^2}{\partial x^2}\frac{\partial u_m^3}{\partial x^3} - \frac{\partial u_m^2}{\partial x^3}\frac{\partial u_m^3}{\partial x^2}\right]\right)\\ &- \frac{\partial}{\partial x^2}\left(u_m^1\left[\frac{\partial u_m^2}{\partial x^1}\frac{\partial u_m^3}{\partial x^3} - \frac{\partial u_m^2}{\partial x^3}\frac{\partial u_m^3}{\partial x^1}\right]\right)\\ &+ \frac{\partial}{\partial x^3}\left(u_m^1\left[\frac{\partial u_m^2}{\partial x^1}\frac{\partial u_m^3}{\partial x^2} - \frac{\partial u_m^2}{\partial x^2}\frac{\partial u_m^3}{\partial x^1}\right]\right).\end{aligned}$$

Thus convergence $\det(\nabla u_m) \to \det(\nabla u)$ in the sense of distributions follows by weak convergence in $L^{3/2}(\Omega')$ of the terms in brackets [- -], proved above, strong convergence $u_m \to u$ in $L^3(\Omega')$, and weak continuity of the distributional derivative. Finally, by uniqueness of the weak limit in the distribution sense, it follows that $\det(\nabla u) = h$ and $\det(\nabla u_m) \rightharpoonup \det(\nabla u)$ weakly in L^1_{loc}, as claimed. □

Observe that in the language of differential forms the divergence structure of a Jacobian or its minors is even more apparent. In fact, for any smooth function $u = (u^1, u^2, u^3)\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ we have

$$\det(\nabla u)\, dx = *\det(\nabla u) = du^1 \wedge du^2 \wedge du^3,$$

where d denotes exterior derivative and where $*$ denotes the Hodge star operator (which in this case converts a function on Ω into a 3-form). Now $dd = 0$, and therefore

$$du^1 \wedge du^2 \wedge du^3 = d(u^1 du^2 \wedge du^3),$$

which immediately implies the asserted divergence structure. Moreover, this result (and therefore Theorem 3.3) generalizes to any dimension n, for $u_m \rightharpoonup u$ weakly in $H^{1,n}_{loc}(\Omega;\mathbb{R}^n)$ with $\det(\nabla u_m) \rightharpoonup h$ weakly in $L^1_{loc}(\Omega)$ as $m \to \infty$.

The assumption $\det(\nabla u_m) \rightharpoonup h \in L^1_{loc}(\Omega)$ at first sight may appear rather awkward. However, examples by Ball-Murat [1] show that weak $H^{1,3}$-convergence in general does not imply weak L^1-convergence of the Jacobian. This difficulty does not arise if we assume weak convergence in $H^{1,3+\varepsilon}$ for some $\varepsilon > 0$.

Hence, adding appropriate growth conditions on W to ensure coerciveness of the functional $\int_\Omega W(\nabla u)\, dx$ on the space $H^{1,3+\varepsilon}(\Omega;\mathbb{R}^3)$ for some $\varepsilon > 0$, from Theorem 3.3 the reader can derive existence theorems for deformations of elastic materials involving polyconvex stored energy functions. As a further reference for such results, see Ciarlet [1] or Dacorogna [1], [2]. Recently, more general results on weak continuity of determinants and corresponding existence theorems in nonlinear elasticity have been obtained by Giaquinta-Modica-Souček [1] and Müller [1], [2], [3].

The regularity theory for problems in nonlinear elasticity is still evolving. Some material can be found in the references cited above. In particular, the

question of cavitation of elastic materials has been studied. See for instance Giaquinta-Modica-Souček [1].

Convergence Results for Nonlinear Elliptic Equations

We close this section with another simple and useful example of how compensated compactness methods may be applied in a nonlinear situation. The following result is essentially "Murat's lemma" from Tartar [2, p. 278]:

3.4 Theorem. *Suppose $u_m \in H_0^{1,2}(\Omega)$ is a sequence of solutions to an elliptic equation*

$$\begin{aligned} -\Delta u_m &= f_m \qquad \text{in } \Omega \\ u_m &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

in a smooth and bounded domain Ω in $\mathbb{R}^n$. Suppose $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega)$ while (f_m) is bounded in $L^1(\Omega)$. Then for a subsequence $m \to \infty$ we have $\nabla u_m \to \nabla u$ in $L^q(\Omega)$ for any $q < 2$, and $\nabla u_m \to \nabla u$ pointwise almost everywhere.

Proof. Choose $p > n$ and let $\varphi_m \in H_0^{1,p}(\Omega)$ satisfy

$$\|\varphi_m\|_{H_0^{1,p}} \le 1$$

$$\int_\Omega (\nabla u_m - \nabla u)\nabla\varphi_m \, dx = \sup_{\varphi \in H_0^{1,p}(\Omega),\ \|\varphi\|_{H_0^{1,p}} \le 1} \int_\Omega (\nabla u_m - \nabla u)\nabla\varphi \, dx \ .$$

By the Calderón-Zygmund inequality in L^p, see Simader[1], the latter

$$\sup_{\varphi \in H_0^{1,p}(\Omega),\ \|\varphi\|_{H_0^{1,p}} \le 1} \int_\Omega (\nabla u_m - \nabla u)\nabla\varphi \, dx \ge c^{-1}\|u_m - u\|_{H_0^{1,q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

On the other hand, by Sobolev's embedding $H_0^{1,p}(\Omega) \hookrightarrow C^{1-\frac{n}{p}}(\overline{\Omega})$. Hence by the Arzéla-Ascoli theorem we may assume that $\varphi_m \rightharpoonup \varphi$ weakly in $H_0^{1,p}(\Omega)$ and uniformly in $\overline{\Omega}$. (See Theorem A.5.) Thus

$$\begin{aligned} \int_\Omega (\nabla u_m - \nabla u)\nabla\varphi_m \, dx &= \int_\Omega (\nabla u_m - \nabla u)(\nabla\varphi_m - \nabla\varphi) \, dx + o(1) \\ &= \lim_{l\to\infty} \int_\Omega (\nabla u_m - \nabla u_l)(\nabla\varphi_m - \nabla\varphi) \, dx + o(1) \\ &= \lim_{l\to\infty} \int_\Omega (f_m - f_l)(\varphi_m - \varphi) \, dx + o(1) \\ &\le 2 \sup_{l\in\mathbb{N}} \|f_l\|_{L^1} \|\varphi_m - \varphi\|_{L^\infty} + o(1) = o(1), \end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$.

It follows that $\nabla u_m \to \nabla u$ in $L^{q_0}(\Omega)$ for some $q_0 \geq 1$. But then, by Hölder's inequality, for any $q < 2$ we have

$$\|\nabla u_m - \nabla u\|_{L^q} \leq \|\nabla u_m - \nabla u\|_{L^1}^{\gamma} \|\nabla u_m - \nabla u\|_{L^2}^{1-\gamma} \to 0\,,$$

where $\frac{1}{q} = \gamma + \frac{1-\gamma}{2}$. □

Results like Theorem 3.4 are needed if one wants to solve nonlinear partial differential equations

$$-\Delta u = f(x, u, \nabla u) \tag{3.3}$$

with quadratic growth

$$|f(x, u, p)| \leq c(1 + |p|^2)$$

by approximation methods. Assuming some uniform control on approximate solutions u_m of (3.3) in $H^{1,2}$, Theorem 3.4 assures that

$$f(x, u_m, \nabla u_m) \to f(x, u, \nabla u) \quad \text{almost everywhere,}$$

where u is the weak limit of a suitable sequence (u_m). Given some further structure conditions on f, then there are various ways of passing to the limit $m \to \infty$ in Equation (3.3); see, for instance, Frehse [2] or Evans [2].

3.5 Example. As a model problem consider the equation

$$A(u) = -\Delta u + u|\nabla u|^2 = h \quad \text{in } \Omega \tag{3.4}$$

$$u = 0 \qquad\qquad \text{on } \partial\Omega \tag{3.5}$$

on a smooth and bounded domain $\Omega \subset \mathbb{R}^n$, with $h \in L^\infty(\Omega)$. This is a special case of a problem studied by Bensoussan-Boccardo-Murat [1; Theorem 1.1, p. 350]. Note that the nonlinear term $g(u, p) = u|p|^2$ satisfies the condition

$$g(u, p)u \geq 0\,. \tag{3.6}$$

Approximate g by functions

$$g_\varepsilon(u, p) = \frac{g(u, p)}{1 + \varepsilon|g(u, p)|}\,, \varepsilon > 0\,,$$

satisfying $|g_\varepsilon| \leq \frac{1}{\varepsilon}$ and $g_\varepsilon(u, p) \cdot u \geq 0$ for all u, p.

Now, since g_ε is uniformly bounded, the map $H_0^{1,2}(\Omega) \ni u \mapsto g_\varepsilon(u, \nabla u) \in H^{-1}(\Omega)$ is compact and bounded for any $\varepsilon > 0$. Denote $A_\varepsilon(u) = -\Delta u + g_\varepsilon(u, \nabla u)$ the perturbed operator A. By Schauder's fixed point theorem, see, for instance, Deimling [1; Theorem 8.8, p. 60], applied to the map $u \mapsto (-\Delta)^{-1}\big(h - g_\varepsilon(u, \nabla u)\big)$ on a sufficiently large ball in $H_0^{1,2}(\Omega)$, there is a solution $u_\varepsilon \in H_0^{1,2}(\Omega)$ of the equation $A_\varepsilon u_\varepsilon = h$ for any $\varepsilon > 0$. In addition, since $g_\varepsilon(u, p) \cdot u \geq 0$ we have

$$\|u_\varepsilon\|^2_{H_0^{1,2}} \le \langle u_\varepsilon, A_\varepsilon u_\varepsilon\rangle = \langle u_\varepsilon, h\rangle \le \|u_\varepsilon\|_{H_0^{1,2}} \|h\|_{H^{-1}} \ ,$$

and (u_ε) is uniformly bounded in $H_0^{1,2}(\Omega)$ for $\varepsilon > 0$. Moreover, since the nonlinear term g_ε satisfies

$$\begin{aligned} g_\varepsilon(u,p) = \frac{u|p|^2}{1+\varepsilon|u|\,|p|^2} &\le \frac{\big(1+|u|^2\big)|p|^2}{1+\varepsilon|u|\,|p|^2} \\ &\le |p|^2 + g_\varepsilon(u,p)u \ , \end{aligned}$$

we also deduce the uniform L^1-bound

$$\begin{aligned} \|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1} &\le \|u_\varepsilon\|^2_{H_0^{1,2}} + \int_\Omega u_\varepsilon g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\, dx \\ &= \langle u_\varepsilon, A_\varepsilon u_\varepsilon\rangle \le c \ . \end{aligned}$$

We may assume that a sequence (u_{ε_m}) as $\varepsilon_m \to 0$ weakly converges in $H_0^{1,2}(\Omega)$ to a limit $u \in H_0^{1,2}(\Omega)$. By Theorem 3.4, moreover, we may assume that $u_m = u_{\varepsilon_m}$ converges strongly in $H_0^{1,q}(\Omega)$ and that u_m and ∇u_m converge pointwise almost everywhere. To show that u weakly solves (3.4), (3.5) we now use the "Fatou lemma technique" of Frehse [2]. As a preliminary step we establish a uniform L^∞-bound for the sequence (u_m).
Multiply the approximate equations by u_m to obtain the differential inequality

$$\begin{aligned} -\Delta\big(\frac{|u_m|^2}{2}\big) &\le -\Delta\big(\frac{|u_m|^2}{2}\big) + |\nabla u_m|^2 + u_m g_{\varepsilon_m}(u_m, \nabla u_m) \\ &= h u_m \le C(\delta) + \delta\frac{|u_m|^2}{2} \ , \end{aligned}$$

for any $\delta > 0$. Choosing $\delta < \lambda_1$, the first eigenvalue of $-\Delta$ on $H_0^{1,2}(\Omega)$, the weak maximum principle implies a uniform bound for u_m in L^∞. (See Theorem B.7 and its application in Appendix B.)

Next, testing the approximate equations $A_{\varepsilon_m}(u_m) = h$ with $\varphi = \xi\exp(\gamma u_m)$, where $\xi \in C_0^\infty(\Omega)$ is non-negative, upon integrating by parts we obtain

$$\begin{aligned} \int_\Omega \Big(\gamma|\nabla u_m|^2 + g_{\varepsilon_m}(u_m, \nabla_{u_m})\Big)\xi\exp(\gamma u_m)\, dx \\ + \int_\Omega \big(\nabla u_m \nabla\xi - h\xi\big)\exp(\gamma u_m)\, dx = 0 \ . \end{aligned}$$

Note that on account of the growth condition

$$|g_{\varepsilon_m}(u,p)| \le |u|\,|p|^2$$

and the uniform bound $\|u_m\|_{L^\infty} \le C_0$ derived above, for $|\gamma| \ge C_0$ the term

$$\gamma|\nabla u_m|^2 + g_{\varepsilon_m}(u_m, \nabla u_m)$$

has the same sign as γ. Moreover, this term converges pointwise almost everywhere to the same expression involving u instead of u_m. Hence, by Fatou's lemma, upon passing to the limit $m \to \infty$ we obtain

$$\langle \xi \exp(\gamma u), A(u) - h \rangle \leq 0$$

if $\gamma \geq C_0$, respectively ≥ 0, if $\gamma \leq -C_0$. This holds for all non-negative $\xi \in C_0^\infty(\Omega)$ and hence also for $\xi \geq 0$ belonging to $H_0^{1,2} \cap L^\infty(\Omega)$. Setting $\xi = \xi_0 \exp(-\gamma u)$ we obtain

$$\langle \xi_0, A(u) - h \rangle \leq 0 \text{ and } \geq 0,$$

for any $\xi_0 \geq 0$, $\xi_0 \in C_0^\infty(\Omega)$. Hence u is a weak solution of (3.4), (3.5), as desired.

More sophisticated variants and applications of Theorem 3.4 are given by Bensoussan-Boccardo-Murat [1], Boccardo-Murat-Puel [1], and Frehse [1], [2].

Hardy Space Methods

The Hardy $\mathcal{H}^p$-spaces play an important role in harmonic analysis. In the theory of partial differential equations, the space $\mathcal{H}^1$ is of particular interest. For instance, the Calderón-Zygmund theorem, asserting that for $1 < p < \infty$ any function $u \in L^p$ with $\Delta u \in L^p$ belongs to $H^{2,p}_{loc}$, ceases to be valid in the limit case $p = 1$. However, the theorem remains true if we substitute L^1 by the slightly smaller space $\mathcal{H}^1$. While the harmonic analysts' definition of $\mathcal{H}^1$ is rather unwieldy and hardly lends itself to applications in the theory of partial differential equations, recently an observation of Müller [2], [3] has led to the discovery of simple criteria for composite functions to belong to $\mathcal{H}^1$. In particular, it was shown by Coifman-Lions-Meyer-Semmes [1] that the Jacobian of a function $u \in H^{1,n}(\mathbb{R}^n; \mathbb{R}^n)$ belongs to this space, and similarly for $l \times l$-sub-determinants of ∇u for $u \in H^{1,l}(\mathbb{R}^n; \mathbb{R}^N)$, for any $l, n, N \in \mathbb{N}$.

As an application, we derive the assertion of Theorem 3.4 in the case that for a sequence $u_m \rightharpoonup u$ weakly in $H^{1,2}(\mathbb{R}^n; \mathbb{R}^N)$ the sequence $-\Delta u_m = f_m$ is bounded in $\mathcal{H}^1$. In fact, by the extension of the Calderón-Zygmund theorem to $\mathcal{H}^1$, the sequence $(\nabla^2 u_m)$ is bounded in $\mathcal{H}^1 \subset L^1$, and therefore, by Rellich's compactness result, $\nabla u_m \to \nabla u$ locally in L^1 as $m \to \infty$. Boundedness of (f_m) in $\mathcal{H}^1$ by the result of Coifman-Lions-Meyer-Semmes [1] is guaranteed if, for instance, for each $m \in \mathbb{N}$ the function f_m is a linear combination of 2×2 sub-determinants of $\nabla F(u_m)$, where $F: \mathbb{R}^N \to \mathbb{R}^L$ is a smooth, bounded function with bounded derivative. For more information about Hardy spaces and their applications, see, for instance, Torchinsky [1] or Semmes [1].

4. The Concentration-Compactness Principle

As we have seen in our analysis of the Plateau problem, Section 2.7, a very serious complication for the direct methods to be applicable arises in the presence of non-compact group actions.

If, in a terminology borrowed from physics which we will try to make more precise later, the action is a "manifest" symmetry – as in the case of the conformal group of the disk acting on Dirichlet's integral for minimal surfaces – we may be able to eliminate the action by a suitable normalization. This was the reason for introducing the three-point condition on admissible functions for the Plateau problem in the proof of Theorem 2.8. However, if the action is "hidden", such a normalization is not possible and there is no hope that *all* minimizing sequences converge to a minimizer. Even worse, the variational problem may not have a solution. For such problems, P.-L. Lions developed his concentration-compactness principle. On the basis of this principle, for many constrained minimization problems it is possible to state necessary and sufficient conditions for the convergence of all minimizing sequences satisfying the given constraint. These conditions involve a delicate comparison of the given functional in variation and a (family of) functionals "at infinity" (on which the group action is manifest).

Rather than dwell on abstract notions we prefer to give an example – a variant of problem (2.1), (2.3) – which will bring out the main ideas immediately.

4.1 Example. Let $a\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function $a > 0$ and suppose that

$$a(x) \to a_\infty > 0 \qquad \text{as } |x| \to \infty \ .$$

We look for positive solutions u of the equation

$$-\Delta u + a(x)u = u|u|^{p-2} \qquad \text{in } \mathbb{R}^n \ , \tag{4.1}$$

decaying at infinity, that is

$$u(x) \to 0 \qquad \text{as } |x| \to \infty \ . \tag{4.2}$$

Here $p > 2$ may be an arbitrary number, if $n = 1, 2$. If $n \geq 3$ we suppose that $p < \frac{2n}{n-2}$. This guarantees that the imbedding

$$H^{1,2}(\Omega) \to L^p(\Omega)$$

is compact for any $\Omega \subset\subset \mathbb{R}^n$.

Note that, as in the proof of Theorem 2.1, solutions of Equation (4.1) correspond to critical points of the functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a(x)|u|^2 \right) dx$$

on $H^{1,2}(\mathbb{R}^n)$, restricted to the unit sphere

$$M = \{u \in H^{1,2}(\mathbb{R}^n) \; ; \; \int_{\mathbb{R}^n} |u|^p \, dx = 1\}$$

in $L^p(\mathbb{R}^n)$. Moreover, if $a(x) \equiv a_\infty$, E is invariant under translations

$$u \mapsto u_{x_0}(x) = u(x - x_0) \; .$$

In general, for any $u \in H^{1,2}(\mathbb{R}^n)$, after a substitution of variables

$$E(u_{x_0}) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a(x + x_0)|u|^2 \right) dx \to \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a_\infty |u|^2 \right) dx$$

as $|x_o| \to \infty$, whence it may seem appropriate to call

$$E^\infty(u) := \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a_\infty |u|^2 \right) dx$$

the *functional at infinity* associated with E. The following result is due to Lions [2; Theorem I.2].

4.2 Theorem. *Suppose*

$$I := \inf_M E < \inf_M E^\infty =: I^\infty \; , \tag{4.3}$$

then there exists a positive solution $u \in H^{1,2}(\mathbb{R}^n)$ of Equation (4.1). Moreover, condition (4.3) is necessary and sufficient for the relative compactness of all minimizing sequences for E in M.

Proof. Clearly, (4.3) is necessary for the relative compactness of all minimizing sequences in M. Indeed, suppose $I^\infty \leq I$ and let (u_m) be a minimizing sequence for E^∞. Then also $(\tilde{u}_m)$, given by $\tilde{u}_m = u_m(\cdot + x_m)$, is a minimizing sequence for E^∞, for any sequence (x_m) in $\mathbb{R}^n$. Choosing $|x_m|$ large enough such that

$$\left| E(\tilde{u}_m) - E^\infty(\tilde{u}_m) \right| \leq \frac{1}{m} \; ,$$

moreover, $(\tilde{u}_m)$ is a minimizing sequence for E. In addition, we can achieve that

$$\tilde{u}_m \to 0 \qquad \text{locally in } \; L^2 \; ,$$

whence $(\tilde{u}_m)$ cannot be relatively compact.

Note that this argument also proves that the weak inequality $I \leq I^\infty$ always holds true, regardless of the particular choice of the function a.

We now show that condition (4.3) is also sufficient for the relative compactness of minimizing sequences. The existence of a positive solution to (4.1) then follows as in the proof of Theorem 2.1.

Let (u_m) be a minimizing sequence for E in M with $E(u_m) \to I$. We may assume that $u_m \rightharpoonup u$ weakly in $L^p(\mathbb{R}^n)$. By continuity, a is uniformly positive on $\mathbb{R}^n$. Hence we also have

$$\|u_m\|^2_{H^{1,2}} \le c\, E(u_m) \le C < \infty\ ,$$

and in addition we may assume that $u_m \rightharpoonup u$ weakly in $H^{1,2}(\mathbb{R}^n)$ and pointwise almost everywhere. Let $u_m = v_m + u$. Observe that the family $(u|u_m - \vartheta u|^{p-1})$ is equi-integrable on $\mathbb{R}^n$, uniformly in $0 \le \vartheta \le 1$. Hence by Vitali's theorem

$$\begin{aligned}\int |u_m|^p\,dx - \int |u_m - u|^p\,dx &= -\iint_0^1 \frac{d}{d\vartheta}|u_m - \vartheta u|^p\,d\vartheta\,dx \\ &= p\iint_0^1 u(u_m - \vartheta u)|u_m - \vartheta u|^{p-2}\,d\vartheta\,dx \\ &\to p\iint_0^1 u(u - \vartheta u)|u - \vartheta u|^{p-2}\,d\vartheta\,dx = \int |u|^p\,dx\ ,\end{aligned} \tag{4.4}$$

where $\int \ldots dx$ denotes integration over $\mathbb{R}^n$; that is,

$$\int_{\mathbb{R}^n} |u|^p\,dx + \int_{\mathbb{R}^n} |v_m|^p\,dx \to 1\ .$$

Similarly, we have

$$\begin{aligned}E(u_m) &= E(v_m + u) \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \big\{\big(|\nabla u|^2 + 2\nabla u\nabla v_m + |\nabla v_m|^2\big) \\ &\quad + a(x)\big(|u|^2 + 2uv_m + |v_m|^2\big)\big\}\,dx \\ &= E(u) + E(v_m) + \int_{\mathbb{R}^n} \big(\nabla u\nabla v_m + a(x)uv_m\big)\,dx,\end{aligned} \tag{4.5}$$

and the last term converges to zero by weak convergence $v_m = u_m - u \rightharpoonup 0$ in $H^{1,2}(\mathbb{R}^n)$.

Moreover, for any $\varepsilon > 0$, letting

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n\ ;\ |a(x) - a_\infty| \ge \varepsilon\} \subset\subset \mathbb{R}^n,$$

since $v_m \to 0$ locally in L^2, the integral

$$\begin{aligned}&\int_{\mathbb{R}^n} (a(x) - a_\infty)|v_m|^2\,dx \\ &\qquad \le \varepsilon \int_{\mathbb{R}^n} |v_m|^2 + \sup_{\mathbb{R}^n} |a(x)| \int_{\Omega_\varepsilon} |v_m|^2\,dx \\ &\qquad \le c\varepsilon + o(1)\ .\end{aligned}$$

Here and in the following, $o(1)$ denotes error terms such that $o(1) \to 0$ as $m \to \infty$. Hence this integral can be made arbitrarily small if we first choose $\varepsilon > 0$ sufficiently small and then let $m \ge m_0(\varepsilon)$ be sufficiently large. That is, we have

$$E(u_m) = E(u) + E^\infty(v_m) + o(1) \ .$$

By homogeneity, if we denote $\lambda = \int_{\mathbb{R}^n} |u|^p \, dx$,

$$E(u) = \lambda^{2/p} E\left(\lambda^{-1/p} u\right) \geq \lambda^{2/p} I, \quad \text{if } \lambda > 0 \ ,$$

$$E^\infty(v_m) = (1-\lambda)^{2/p} E^\infty\left((1-\lambda)^{-1/p} v_m\right) \geq (1-\lambda)^{2/p} I^\infty + o(1), \text{ if } \lambda < 1 \ .$$

Hence, for all $\lambda \in [0,1]$, we obtain the estimate

$$\begin{aligned} I &= E(u_m) + o(1) = E(u) + E^\infty(v_m) + o(1) \\ &\geq \lambda^{2/p} I + (1-\lambda)^{2/p} I^\infty + o(1) \\ &\geq \left(\lambda^{2/p} + (1-\lambda)^{2/p}\right) I + o(1) \ . \end{aligned}$$

Since $p > 2$ this implies that $\lambda \in \{0,1\}$. But if $\lambda = 0$, we obtain that

$$I \geq I^\infty + o(1) > I$$

for large m; a contradiction.

Therefore $\lambda = 1$; that is, $u_m \to u$ in L^p, and $u \in M$. By convexity of E, moreover,

$$E(u) \leq \liminf_{m\to\infty} E(u_m) = I \ ,$$

and u minimizes E in M. Hence also $E(u_m) \to E(u)$. Finally, by (4.5)

$$\begin{aligned} \|u_m - u\|^2_{H^{1,2}} &\leq cE(u_m - u) \\ &= c\left(E(u_m) - E(u)\right) + o(1) \to 0 \ , \end{aligned}$$

and $u_m \to u$ strongly in $H^{1,2}(\mathbb{R}^n)$. The proof is complete. □

Regarding $|u_m|^p \, dx$ as a measure on $\mathbb{R}^n$, a systematic approach to such problems is possible via the following lemma (P.L. Lions [1; p. 115 ff.]).

4.3 Concentration-Compactness Lemma I . *Suppose μ_m is a sequence of probability measures on $\mathbb{R}^n$: $\mu_m \geq 0$, $\int_{\mathbb{R}^n} d\mu_m = 1$. There is a subsequence (μ_m) such that one of the following three conditions holds:*
(1°) (Compactness) There exists a sequence $x_m \subset \mathbb{R}^n$ such that for any $\varepsilon > 0$ there is a radius $R > 0$ with the property that

$$\int_{B_R(x_m)} d\mu_m \geq 1 - \varepsilon$$

for all m.
(2°) (Vanishing) For all $R > 0$ there holds

$$\lim_{m\to\infty} \left(\sup_{x\in\mathbb{R}^n} \int_{B_R(x)} d\mu_m \right) = 0 \ .$$

(3°) (Dichotomy) There exists a number λ, $0 < \lambda < 1$, such that for any $\varepsilon > 0$ there is a number $R > 0$ and a sequence (x_m) with the following property: Given $R' > R$ there are non-negative measures μ_m^1, μ_m^2 such that

$$
\begin{aligned}
&0 \le \mu_m^1 + \mu_m^2 \le \mu_m \ , \\
&\operatorname{supp}(\mu_m^1) \subset B_R(x_m), \ \operatorname{supp}(\mu_m^2) \subset \mathbb{R}^n \setminus B_{R'}(x_m) \ , \\
&\limsup_{m\to\infty} \left(\left| \lambda - \int_{\mathbb{R}^n} d\mu_m^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^n} d\mu_m^2 \right| \right) \le \varepsilon \ .
\end{aligned}
$$

Proof. The proof is based on the notion of concentration function

$$
Q(r) = \sup_{x\in\mathbb{R}^n} \left(\int_{B_r(x)} d\mu \right)
$$

of a non-negative measure, introduced by Lévy [1].

Let Q_m be the concentration functions associated with μ_m. Note that (Q_m) is a sequence of non-decreasing, non-negative bounded functions on $[0, \infty[$ with $lim_{R\to\infty} Q_m(R) = 1$. Hence, (Q_m) is locally bounded in BV on $[0, \infty[$ and there exists a subsequence (μ_m) and a bounded, non-negative, non-decreasing function Q such that

$$
Q_m(R) \to Q(R) \qquad (m \to \infty) \ ,
$$

for almost every $R > 0$. We normalize Q to be continuous from the left. Since Q_m is non-decreasing, this then also implies that for every $R > 0$ we have

$$
Q(R) \le \liminf_{m\to\infty} Q_m(R).
$$

Let

$$
\lambda = \lim_{R\to\infty} Q(R) \ .
$$

Clearly $0 \le \lambda \le 1$. If $\lambda = 0$, we have "vanishing", case (2°). Suppose $\lambda = 1$. Then for some $R_0 > 0$ we have $Q(R_0) > \frac{1}{2}$. For any $m \in \mathbb{N}$ let x_m satisfy

$$
Q_m(R_0) \le \int_{B_{R_0}(x_m)} d\mu_m \ + \frac{1}{m} \ .
$$

Now for $0 < \varepsilon < \frac{1}{2}$ fix R such that $Q(R) > 1 - \varepsilon > \frac{1}{2}$ and let y_m satisfy

$$
Q_m(R) \le \int_{B_R(y_m)} d\mu_m \ + \frac{1}{m} \ .
$$

Then, with error $o(1) \to 0$ as $m \to \infty$, we have

$$
\begin{aligned}
\int_{B_R(y_m)} d\mu_m \ + \int_{B_{R_0}(x_m)} d\mu_m \ &\ge Q_m(R_0) + Q_m(R) + o(1) \\
&\ge Q(R_0) + Q(R) + o(1)
\end{aligned}
$$

and the right-hand side is

$$> 1 = \int_{\mathbb{R}^n} d\mu_m$$

for sufficiently large m. It follows that for such m

$$B_R(y_m) \cap B_{R_0}(x_m) \neq \emptyset \ .$$

That is, $B_R(y_m) \subset B_{2R+R_0}(x_m)$ and hence

$$1 - \varepsilon \leq \int_{B_{2R+R_0}(x_m)} d\mu_m$$

for large m. Choosing R even larger, if necessary, we can achieve that (1°) holds for all m.

If $0 < \lambda < 1$, given $\varepsilon > 0$ choose R and a sequence (x_m) – depending on ε and R – such that

$$Q_m(R) \geq \int_{B_R(x_m)} d\mu_m > \lambda - \varepsilon \ ,$$

if $m \geq m_0(\varepsilon)$. Enlarging $m_0(\varepsilon)$, if necessary, we can also find a sequence $R_m \to \infty$ such that

$$Q_m(R) \leq Q_m(R_m) < \lambda + \varepsilon \ ,$$

if $m \geq m_0(\varepsilon)$. Moreover, given $R' > R$, we may assume that $R_m \geq R'$ for all m. Now let $\mu_m^1 = \mu_m \llcorner B_R(x_m)$, the restriction of μ_m to $B_R(x_m)$. Similarly, define $\mu_m^2 = \mu_m \llcorner \big(\mathbb{R}^n \setminus B_{R_m}(x_m)\big)$. Obviously

$$0 \leq \mu_m^1 + \mu_m^2 \leq \mu_m \ ,$$

and

$$\operatorname{supp}(\mu_m^1) \subset B_R(x_m), \ \operatorname{supp}(\mu_m^2) \subset \mathbb{R}^n \setminus B_{R_m}(x_m) \subset \mathbb{R}^n \setminus B_{R'}(x_m) \ .$$

Finally, for $m \geq m_0(\varepsilon)$ we can estimate

$$\begin{aligned} &\left|\lambda - \int_{\mathbb{R}^n} d\mu_m^1\right| + \left|1 - \lambda - \int_{\mathbb{R}^n} d\mu_m^2\right| \\ &\qquad = \left|\lambda - \int_{B_R(x_m)} d\mu_m\right| + \left|\int_{B_{R_m}(x_m)} d\mu_m - \lambda\right| < 2\varepsilon \ , \end{aligned}$$

which concludes the proof. □

In the context of Theorem 4.2, Lemma 4.3 may be applied to $\mu_m = |u_m|^p \, dx$, $m \in \mathbb{N}$. Dichotomy in this case is made explicit in (4.4). In view of the compactness of the embedding $H^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ on bounded domains Ω for all $p < \frac{2n}{n-2}$ the situation dealt with in Example 4.1 is referred to as the *locally compact case*.

Further complications arise in the presence of non-compact symmetry groups acting locally; for instance, in the case of conformal invariance or invariance under scaling.

Existence of Extremal Functions for Sobolev Embeddings

A typical example is the case of Sobolev's embedding on a (possibly unbounded) domain $\Omega \subset \mathbb{R}^n$.

4.4 Sobolev embeddings. For $u \in C_0^\infty(\Omega),\ k \geq 1,\ p \geq 1$, let

$$\|u\|_{D^{k,p}}^p = \sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p \, dx \ ,$$

and let $D^{k,p}(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ in this norm. Suppose $kp < n$.

By Sobolev's embedding, $D^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, and there exists a (maximal) constant $S = S(\Omega) = S(\Omega, k, n, p)$ such that

$$S\|u\|_{L^q}^p \leq \|u\|_{D^{k,p}}^p, \text{ for all } u \in D^{k,p}(\Omega) \ . \tag{4.6}$$

Using Schwarz-symmetrization (see for instance Polya-Szegö [1; Note A.5, p. 189 ff.]), for $k = 1$, best constants and extremal functions (on $\Omega = \mathbb{R}^n$) can be computed classically, see Talenti [1]; the earliest result in this regard seems to be due to Rodemich[1]. But for $k > 1$ this method can no longer be applied. Using the concentration-compactness principle, however, the existence of extremal functions for Sobolev's embedding can be established in general; see Theorem 4.9 below.

First we note an important property of the embedding (4.6).

4.5 Scale invariance. By invariance of the norms in $D^{k,p}(\mathbb{R}^n)$, respectively $L^q(\mathbb{R}^n)$, under translation and scaling

$$u \mapsto u_R(x) = R^{-n/q} u(x/R) \tag{4.7}$$

the Sobolev constant S is independent of Ω. Indeed, for any domain Ω, extending a function $u \in C_0^\infty(\Omega)$ by 0 outside Ω, we may regard $C_0^\infty(\Omega)$ as a subset of $C_0^\infty(\mathbb{R}^n)$. Similarly, we may regard $D^{k,p}(\Omega)$ as a subset of $D^{k,p}(\mathbb{R}^n)$. Hence we have

$$\begin{aligned} S(\Omega) &= \inf\big\{ \|u\|_{D^{k,p}}^p \ ;\ u \in D^{k,p}(\Omega),\ \|u\|_{L^q} = 1 \big\} \\ &\geq S(\mathbb{R}^n) \ . \end{aligned}$$

Conversely, if $u_m \in D^{k,p}(\mathbb{R}^n)$ is a minimizing sequence for $S(\mathbb{R}^n)$ with $\|u_m\|_{L^q} = 1$, by density of $C_0^\infty(\mathbb{R}^n)$ in $D^{k,p}(\mathbb{R}^n)$ we may assume that $u_m \in C_0^\infty(\mathbb{R}^n)$. After translation, moreover, we have $0 \in \Omega$. Scaling with (4.7), for sufficiently small R_m we can achieve that $v_m = (u_m)_{R_m} \in C_0^\infty(\Omega)$. But by invariance of $\|\cdot\|_{D^{k,p}}, \|\cdot\|_{L^q}$ under (4.7) there now results

$$S(\Omega) \leq \liminf_{m\to\infty} \|v_m\|_{D^{k,p}}^p = S(\mathbb{R}^n) \ ,$$

and $S(\Omega) = S(\mathbb{R}^n) = S$, as claimed.

4.6 The case $k = 1$. For $k = 1$ the Sobolev inequality has an underlying geometric meaning which allows one to analyze this case completely. Consider first the case $p = 1$. For $u \in D^{1,1}(\mathbb{R}^n)$, or even $u \in BV(\mathbb{R}^n)$, $q_1 = \frac{n}{n-1}$, we claim

$$\left(\int_{\mathbb{R}^n} |u|^{q_1}\, dx\right)^{1/q_1} \leq \frac{1}{n^{1/q_1}\omega_{n-1}^{1/n}} \int_{\mathbb{R}^n} |\nabla u|\, dx\ , \tag{4.8}$$

where ω_{n-1} denotes the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$. Observe that equality holds if and only if $u \in BV(\mathbb{R}^n)$ is a scalar multiple of the characteristic function of a ball in $\mathbb{R}^n$. This reflects the fact that

$$K(n) = \frac{1}{n^{1/q_1}\omega_{n-1}^{1/n}} = \sup_{\Omega \subset\subset \mathbb{R}^n} \frac{\big(\mathcal{L}^n(\Omega)\big)^{1/q_1}}{P(\Omega; \mathbb{R}^n)}$$

equals the isoperimetric constant in $\mathbb{R}^n$, and $K(n)$ is achieved if and only if Ω is a ball in $\mathbb{R}^n$. (The perimeter $P(\Omega; \mathbb{R}^n)$ was defined in Theorem 1.4.)

The following proof of (4.8), based on Talenti [2; p. 404], reveals the deep relation between isoperimetric inequalities and best constants for Sobolev embeddings more clearly. (See also Cianchi [1; Lemma 1].)

Proof of (4.8). For $u \in C_0^\infty(\mathbb{R}^n)$, and $t \geq 0$ let

$$\Omega(t) = \{x \in \mathbb{R}^n\ ;\ |u(x)| > t\}\ .$$

Then

$$|u(x)| = \int_0^\infty \chi_{\Omega(t)}(x)\, dt$$

for almost every $x \in \Omega$, and hence by Minkowsky's inequality

$$\begin{aligned} \|u\|_{L^{q_1}} &\leq \int_0^\infty \|\chi_{\Omega(t)}\|_{L^{q_1}}\, dt \\ &= \int_0^\infty \mathcal{L}^n\big(\Omega(t)\big)^{1/q_1}\, dt \\ &\leq K(n) \int_0^\infty P\big(\Omega(t); \mathbb{R}^n\big)\, dt\ . \end{aligned}$$

Finally, by the co-area formula (see for instance Federer [1; Theorem 3.2.11] or Giusti [1; Theorem 1.23])

$$\int_{\mathbb{R}^n} |\nabla u|\, dx = \int_{\mathbb{R}^n} \big|\nabla |u|\big|\, dx = \int_0^\infty P\big(\Omega(t); \mathbb{R}^n\big)\, dt\ ,$$

and (4.8) follows. □

From (4.8) the general case $p \geq 1$ can be derived by applying Hölder's inequality. Denote $q = \frac{np}{n-p} = sq_1$, where $s = \frac{np-p}{n-p} \geq 1$. Then for $u \in C_0^\infty(\mathbb{R}^n)$ we can estimate

$$\begin{aligned}\|u\|_{L^q} = \big\||u|^s\big\|_{L^{q_1}}^{1/s} &\le \big(K(n)\big)^{1/s}\left(\int_{\mathbb{R}^n} |\nabla|u|^s|\,dx\right)^{1/s}\\ &\le \big(sK(n)\big)^{1/s}\left(\int_{\mathbb{R}^n} |\nabla u|\,|u|^{s-1}\,dx\right)^{1/s}\\ &\le \big(sK(n)\big)^{1/s}\|u\|_{D^{1,p}}^{1/s}\|u\|_{L^q}^{1-1/s}\end{aligned}$$

and

$$\|u\|_{L^q} \le sK(n)\|u\|_{D^{1,p}}\ . \tag{4.9}$$

(We do not claim that this constant is sharp.)

4.7 Bounded domains. In contrast to the case $p = 1$, for $p > 1$ the best constant in inequality (4.9) is never achieved on any domain Ω different from $\mathbb{R}^n$; in particular, it is never achieved on a bounded domain. Indeed, if $u \in D^{1,p}(\Omega)$ achieves $S = S(\mathbb{R}^n)$, a multiple of u weakly solves the equation

$$-\nabla\big(|\nabla u|^{p-2}\nabla u\big) = u|u|^{q-2} \qquad \text{in } \mathbb{R}^n \tag{4.10}$$

and vanishes on $\mathbb{R}^n \setminus \Omega$, which contradicts the strong maximum principle for Equation (4.10); see, for instance, Tolksdorf [1].

Of course, we suspect invariance under scaling (4.7) to be responsible for this defect. Note that for any $u \in D^{k,p}(\mathbb{R}^n)$ there holds

$$u_R \rightharpoonup 0 \text{ weakly in } D^{k,p}(\mathbb{R}^n) \text{ as } R \to 0\ ,$$

while S is invariant under scaling. Hence, relative compactness of minimizing sequences cannot be expected. Observe, moreover, that for $u \in C_0^\infty(\mathbb{R}^n)$ the support of u_R lies in a fixed compact set for all $R \le 1$. That is, we encounter a new type of loss of compactness compared to Example 4.1; we are dealing with a problem which is also locally non-compact.

This is the setting for the second concentration-compactness lemma from P.L. Lions [3; Lemma I.1]. Denote

$$\sum_{|\alpha|=k} |D^\alpha u|^p = |D^k u|^p\ ,$$

for convenience.

4.8 Concentration-Compactness Lemma II. *Let $k \in \mathbb{N}$, $p \ge 1$, $kp < n$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. Suppose $u_m \rightharpoonup u$ weakly in $D^{k,p}(\mathbb{R}^n)$ and $\mu_m = |\nabla^k u_m|^p\,dx \rightharpoonup \mu$, $\nu_m = |u_m|^q\,dx \rightharpoonup \nu$ weakly in the sense of measures where μ and ν are bounded non-negative measures on $\mathbb{R}^n$.*
Then we have:
(1°) There exists some at most countable set J, a family $\{x^{(j)}\ ;\ j \in J\}$ of distinct points in $\mathbb{R}^n$, and a family $\{\nu^{(j)}\ ;\ j \in J\}$ of positive numbers such that

$$\nu = |u|^q\,dx + \sum_{j\in J} \nu^{(j)}\delta_{x^{(j)}}\ ,$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^n$.
(2°) In addition we have

$$\mu \geq |\nabla^k u|^p\,dx + \sum_{j\in J} \mu^{(j)}\delta_{x^{(j)}}$$

for some family $\{\mu^{(j)}\ ;\ j \in J\},\ \mu^{(j)} > 0$ satisfying

$$S\left(\nu^{(j)}\right)^{p/q} \leq \mu^{(j)}\ , \qquad \text{for all } j \in J\ .$$

In particular, $\sum_{j\in J}\left(\nu^{(j)}\right)^{p/q} < \infty$.

Proof. Let $v_m = u_m - u \in D^{k,p}(\mathbb{R}^n)$. Then $v_m \rightharpoonup 0$ weakly in $D^{k,p}$, and by (4.4) we have

$$\begin{aligned}\omega_m := \nu_m - |u|^q\,dx &= \left(|u_m|^q - |u|^q\right)dx \\ &= |u_m - u|^q\,dx + o(1) = |v_m|^q\,dx + o(1)\ ,\end{aligned}$$

where $o(1) \rightharpoonup 0$ as $m \to \infty$. Also let $\lambda_m := |\nabla^k v_m|^p\,dx$. We may assume that $\lambda_m \rightharpoonup \lambda$, while $\omega_m \rightharpoonup \omega = \nu - |u|^q\,dx$ weakly in the sense of measures, where $\lambda, \omega \geq 0$.

Choose $\xi \in C_0^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi|^q\,d\omega &= \lim_{m\to\infty}\int_{\mathbb{R}^n} |\xi|^q\,d\omega_m = \lim_{m\to\infty}\int_{\mathbb{R}^n} |v_m\,\xi|^q\,dx \\ &\leq S^{-q/p}\ \liminf_{m\to\infty}\left(\int_{\mathbb{R}^n} |\nabla^k(v_m\xi)|^p\,dx\right)^{q/p} \\ &= S^{-q/p}\ \liminf_{m\to\infty}\left(\int_{\mathbb{R}^n} |\xi|^p|\nabla^k v_m|^p\,dx\right)^{q/p} \\ &= S^{-q/p}\left(\int_{\mathbb{R}^n} |\xi|^p\,d\lambda\right)^{q/p}\ .\end{aligned}$$

Observe that by Rellich's theorem any lower order terms like $|\nabla^l\xi||\nabla^{k-l}v_m| \to 0$ in L^p, as $m \to \infty$. That is, there holds

$$S\left(\int_{\mathbb{R}^n} |\xi|^q\,d\omega\right)^{p/q} \leq \int_{\mathbb{R}^n} |\xi|^p\,d\lambda \tag{4.11}$$

for all $\xi \in C_0^\infty(\mathbb{R}^n)$. Now let $\{x^{(j)}\ ;\ j \in J\}$ be the atoms of the measure ω and decompose $\omega = \omega_0 + \sum_{j\in J}\nu^{(j)}\delta_{x^{(j)}}$, with ω_0 free of atoms. Since $\int_{\mathbb{R}^n} d\omega < \infty$, J is an at most countable set. Moreover, $\omega_0 \geq 0$. Choosing ξ such that $0 \leq \xi \leq 1,\ \xi\left(x^{(j)}\right) = 1$, from (4.11) we see that

$$\lambda \geq S\big(\nu^{(j)}\big)^{p/q}\delta_{x^{(j)}}, \qquad \text{for all } j \in J \ .$$

Since $|\nabla^k u_m|^p - |\nabla^k v_m|^p$ is of lower order than $|\nabla^k v_m|^p$ at points of concentration, the latter estimate also holds for μ.

On the other hand, by weak lower semi-continuity we have

$$\mu \geq |\nabla^k u|^p \, dx \ .$$

The latter measure and the measures $\delta_{x^{(j)}}$ being relatively singular, (2°) follows.

Now, for any open set $\Omega \subset \mathbb{R}^n$ such that $\int_\Omega d\lambda \leq S$, by (4.11) with $\xi = \xi_k \in C_0^\infty(\Omega)$ converging to the characteristic function of Ω as $k \to \infty$, we have

$$\int_\Omega d\omega \leq \left(\int_\Omega d\omega\right)^{p/q} \leq S^{-1}\int_\Omega d\lambda \leq 1 \ . \tag{4.12}$$

That is, ω is absolutely continuous with respect to λ and by the Radon-Nikodym theorem there exists $f \in L^1(\mathbb{R}^n;\lambda)$ such that $d\omega = f\,d\lambda$, λ-almost everywhere. Moreover, for λ-almost every $x \in \mathbb{R}^n$ we have

$$f(x) = \lim_{\rho\to 0}\left(\frac{\int_{B_\rho(x)} d\omega}{\int_{B_\rho(x)} d\lambda}\right) \ .$$

But then by (4.12), if x is not an atom of λ,

$$S\,f(x)^{p/q} = \lim_{\rho\to 0}\left(\frac{S\left(\int_{B_\rho(x)} d\omega\right)^{p/q}}{\left(\int_{B_\rho(x)} d\lambda\right)^{p/q}}\right) \leq \lim_{\rho\to 0}\left(\int_{B_\rho(x)} d\lambda\right)^{\frac{q-p}{q}} = 0 \ ,$$

λ-almost everywhere. Since λ has only countably many atoms and ω_0 has no atoms this implies that $\omega_0 = 0$, that is, (1°). □

Finally, we can state the following result; see P.L. Lions [3; Theorem I.1]:

4.9 Theorem. *Let $k \in \mathbb{N}$, $p > 1$, $kp < n$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. Suppose (u_m) is a minimizing sequence for S in $D^{k,p} = D^{k,p}(\mathbb{R}^n)$ with $\|u_m\|_{L^q} = 1$. Then (u_m) up to translation and dilation is relatively compact in $D^{k,p}$.*

Proof. Choose $\tilde{x}_m \in \mathbb{R}^n$, $\tilde{R}_m > 0$ such that for the rescaled sequence

$$v_m(x) = \tilde{R}_m^{-n/q} u_m\left(\frac{x - \tilde{x}_m}{\tilde{R}_m}\right)$$

there holds

$$Q_m(1) = \sup_{x\in\mathbb{R}^n}\int_{B_1(x)} |v_m|^q \, dx = \int_{B_1(0)} |v_m|^q \, dx = \frac{1}{2} \ . \tag{4.13}$$

Since $p > 1$ we may assume that $v_m \rightharpoonup v$ weakly in $L^q(\mathbb{R}^n)$ and weakly in $D^{k,p}(\mathbb{R}^n)$. Consider the families of measures

$$\mu_m = |\nabla^k v_m|^p \, dx, \; \nu_m = |v_m|^q \, dx$$

and apply Lemma 4.3 to the sequence (ν_m). Vanishing is ruled out by our above normalization. If we have dichotomy, let $\lambda \in]0,1[$ be as in Lemma 4.3.(3°) and for $\varepsilon > 0$ determine $R > 0$, a sequence (x_m), and measures $\nu_m^1, \; \nu_m^2$ as in that lemma such that

$$\begin{aligned} &0 \le \nu_m^1 + \nu_m^2 \le \nu_m \; , \\ &\operatorname{supp}(\nu_m^1) \subset B_R(x_m), \; \operatorname{supp}(\nu_m^2) \subset \mathbb{R}^n \setminus B_{2R}(x_m) \; , \\ &\limsup_{m\to\infty} \left\{ \left| \int_{\mathbb{R}^n} d\nu_m^1 - \lambda \right| + \left| \int_{\mathbb{R}^n} d\nu_m^2 - (1-\lambda) \right| \right\} \le \varepsilon \; . \end{aligned}$$

Choosing a sequence $\varepsilon_m \to 0$ with corresponding $R_m > 0$ and x_m, upon passing to a subsequence (ν_m) if necessary, we can achieve that

$$\operatorname{supp}(\nu_m^1) \subset B_{R_m}(x_m), \; \operatorname{supp}(\nu_m^2) \subset \mathbb{R}^n \setminus B_{2R_m}(x_m)$$

and

$$\limsup_{m\to\infty} \left\{ \left| \int_{\mathbb{R}^n} d\nu_m^1 - \lambda \right| + \left| \int_{\mathbb{R}^n} d\nu_m^2 - (1-\lambda) \right| \right\} = 0 \; .$$

Moreover, in view of Lemma 4.3, we may suppose that $R_m \to \infty$ $(m \to \infty)$.

Choose $\varphi \in C_0^\infty(B_2(0))$ with $0 \le \varphi \le 1$ and such that $\varphi \equiv 1$ in $B_1(0)$. For $m \in \mathbb{N}$ let $\varphi_m(x) = \varphi\left(\frac{x - x_m}{R_m}\right)$. Since $p \ge 1$ there holds

$$|\nabla^k v_m|^p \ge |\nabla^k v_m|^p (\varphi_m^p + (1-\varphi_m)^p).$$

By Minkowski's inequality we have

$$\|(\nabla^k v_m)\varphi_m\|_{L^p(\mathbb{R}^n)} \ge \|\nabla^k (v_m \varphi_m)\|_{L^p(\mathbb{R}^n)} - C \sum_{l<k} \|\nabla^l v_m \nabla^{k-l} \varphi_m\|_{L^p(\mathbb{R}^n)},$$

and similarly for $(1-\varphi_m)$ instead of φ_m. Let A_m denote the annulus $A_m = B_{2R_m}(x_m) \setminus B_{R_m}(x_m)$. By Young's inequality then for any $\delta > 0$ we obtain

$$\|\nabla^k v_m\|_{L^p(\mathbb{R}^n)}^p \ge \|\nabla^k (v_m \varphi_m)\|_{L^p(\mathbb{R}^n)}^p + \|\nabla^k (v_m (1-\varphi_m))\|_{L^p(\mathbb{R}^n)}^p - \beta_m,$$

where the error terms β_m can be estimated

$$\beta_m \le \delta \|\nabla^k v_m\|_{L^p(\mathbb{R}^n)}^p + C(\delta) \sum_{l<k} \|\nabla^l v_m \nabla^{k-l} \varphi_m\|_{L^p(A_m)}^p \; .$$

Since $|\nabla^{k-l} \varphi_m| \le C R_m^{l-k}$, by interpolation (as in Adams [1; Theorem 4.14]) we can bound

$$\begin{aligned} \left\| |\nabla^l v_m| \, |\nabla^{k-l} \varphi_m| \right\|_{L^p(A_m)} &\le C \, R_m^{l-k} \|\nabla^l v_m\|_{L^p(A_m)} \\ &\le C \, K \, \gamma \|\nabla^k v_m\|_{L^p(A_m)} + C \, K \, R_m^{-k} \gamma^{-\frac{l}{k-l}} \|v_m\|_{L^p(A_m)} \; . \end{aligned} \tag{4.14}$$

Here γ can be chosen arbitrarily in $]0,1]$, while the constant K depends only on k and n. (Note that estimate (4.14) is invariant under dilation.) Moreover, by Hölder's inequality

$$\begin{aligned} R_m^{-k}\|v_m\|_{L^p(A_m)} &\le R_m^{-k}\big(\mathcal{L}^n(A_m)\big)^{\frac{1}{p}-\frac{1}{q}}\|v_m\|_{L^q(A_m)} \le C\|v_m\|_{L^q(A_m)} \\ &\le C\left[\int_{\mathbb{R}^n} d\nu_m - \left(\int_{\mathbb{R}^n} d\nu_m^1 + \int_{\mathbb{R}^n} d\nu_m^2\right)\right]^{\frac{1}{q}} . \end{aligned}$$

Hence this term tends to 0 as $m \to \infty$, while $\|\nabla^k v_m\|^p_{L^p(A_m)} \le \|v_m\|^p_{D^{k,p}}$ remains uniformly bounded. Choosing a suitable sequence $\gamma = \gamma_m \to 0$, from (4.14) we thus obtain that $\beta_m \le o(1)$, where $o(1) \to 0$ $(m \to \infty)$. Also letting $\delta = \delta_m \to 0$ suitably, by Sobolev's inequality we find

$$\begin{aligned} \|v_m\|^p_{D^{k,p}} &\ge \|v_m\varphi_m\|^p_{D^{k,p}} + \|v_m(1-\varphi_m)\|^p_{D^{k,p}} + o(1) \\ &\ge S\left(\|v_m\varphi_m\|^p_{L^q} + \|v_m(1-\varphi_m)\|^p_{L^q}\right) + o(1) \\ &\ge S\left[\left(\int_{B_{R_m}(x_m)} d\nu_m\right)^{p/q} + \left(\int_{\mathbb{R}^n\setminus B_{2R_m}(x_m)} d\nu_m\right)^{p/q}\right] + o(1) \\ &\ge S\left[\left(\int_{\mathbb{R}^n} d\nu_m^1\right)^{p/q} + \left(\int_{\mathbb{R}^n} d\nu_m^2\right)^{p/q}\right] + o(1) \\ &\ge S\left(\lambda^{p/q} + (1-\lambda)^{p/q}\right) - o(1) , \end{aligned}$$

where $o(1) \to 0$ $(m \to \infty)$. But for $0 < \lambda < 1$ and $p < q$ we have $\lambda^{p/q} + (1-\lambda)^{p/q} > 1$, contradicting the initial assumption that $\|v_m\|^p_{D^{k,p}} = \|u_m\|^p_{D^{k,p}} \to S$. There remains the case $\lambda = 1$, that is, case (1°) of Lemma 4.3.

Let x_m be as in that lemma and for $\varepsilon > 0$ choose $R = R(\varepsilon)$ such that

$$\int_{B_R(x_m)} d\nu_m \ge 1 - \varepsilon .$$

If $\varepsilon < \frac{1}{2}$ our normalization condition (4.13) implies $B_R(x_m) \cap B_1(0) \ne \emptyset$. Hence the conclusion of Lemma 4.3.(1°) also holds with $x_m = 0$, replacing $R(\varepsilon)$ by $2R(\varepsilon) + 1$ if necessary. Thus, if $\nu_m \rightharpoonup \nu$ weakly, it follows that

$$\int_{\mathbb{R}^n} d\nu = 1 .$$

By Lemma 4.8 we may assume that

$$\begin{aligned} \mu_m &\rightharpoonup \mu \ge |\nabla^k v|^p \, dx + \sum_{j\in\mathcal{J}} \mu^{(j)}\delta_{x^{(j)}} \\ \nu_m &\rightharpoonup \nu = |v|^q \, dx + \sum_{j\in\mathcal{J}} \nu^{(j)}\delta_{x^{(j)}} \end{aligned}$$

for certain points $x^{(j)} \in \mathbb{R}^n$, $j \in \mathcal{J}$, and positive numbers $\mu^{(j)}, \nu^{(j)}$ satisfying

$$S\big(\nu^{(j)}\big)^{p/q} \le \mu^{(j)} , \qquad \text{for all } j \in \mathcal{J} .$$

By Sobolev's inequality then

$$\begin{aligned} S + o(1) = \|v_m\|^p_{D^{k,p}} = \int_{\mathbb{R}^n} d\mu_m &\ge \|v\|^p_{D^{k,p}} + \sum_{j\in\mathcal{J}} \mu^{(j)} + o(1) \\ &\ge S \left(\|v\|^{p/q}_{L^q} + \sum_{j\in\mathcal{J}} \big(\nu^{(j)}\big)^{p/q} \right) + o(1) \end{aligned}$$

where $o(1) \to 1$ $(m \to \infty)$. By strict concavity of the map $\lambda \to \lambda^{p/q}$ now the latter will be

$$\begin{aligned} &\ge S \left(\|v\|^q_{L^q} + \sum_{j\in\mathcal{J}} \nu^{(j)} \right)^{p/q} + o(1) \\ &= S \left(\int_{\mathbb{R}^n} d\nu \right)^{p/q} + o(1) = S + o(1) \end{aligned} \tag{4.15}$$

and equality holds if and only if at most one of the terms $\|v\|_{L^q}$, $\nu^{(j)}$, $j \in \mathcal{J}$, is different from 0.

Note that our normalization (4.13) assures that

$$\nu^{(j)} \le \frac{1}{2} \qquad \text{for all } j \in \mathcal{J} .$$

Hence all $\nu^{(j)}$ must vanish, $\|v\|_{L^q} = 1$, and $v_m \to v$ strongly in $L^q(\mathbb{R}^n)$. But then by Sobolev's inequality $\|v\|^p_{D^{k,p}} \ge S$ and $\|v_m\|_{D^{k,p}} \to \|v\|_{D^{k,p}}$ as $m \to \infty$. It follows that $v_m \to v$ in $D^{k,p}(\mathbb{R}^n)$, as desired. The proof is complete. □

As a consequence we obtain

4.10 Corollary. *For any $k \in \mathbb{N}$, any $p > 1$ such that $kp < n$ there exists a function $u \in D^{k,p}(\mathbb{R}^n)$ with $\|u\|_{L^q(\mathbb{R}^n)} = 1$ and $\|u\|_{D^{k,p}(\mathbb{R}^n)} = S$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ and where $S = S(k,p,n)$ is the Sobolev constant.*

Observe that, since Lemma 4.8 requires weak convergence $u_m \rightharpoonup u$ in $D^{k,p}(\mathbb{R}^n)$ the above proof of Theorem 4.9 cannot be extended to the case $p = 1$. In fact, Corollary 4.10 is false in that case, as we have seen that the best constant for Sobolev's embedding $D^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$ is attained (precisely) on characteristic functions of balls; that is, in $BV_{loc}(\mathbb{R}^n)$.

4.11 Notes. (1°) The limiting case $kp = n$ for Sobolev's embedding behaves strikingly different from the case $kp < n$ studied above. Consider $k = 1$ for simplicity. By Sobolev's embedding $W^{k,\frac{n}{k}} \hookrightarrow W^{1,n}$ the results for $k = 1$ extend to the general case. However, for $k > 1$ the results that follow can be slightly improved, see Brezis-Waigner [1] or Ziemer [1].

By results of Trudinger [1] and Moser [3], for any $\Omega \subset\subset \mathbb{R}^n$ the space $W_0^{1,n}(\Omega)$ embeds into the Orlicz space of functions $u\colon \Omega \to \mathbb{R}$ such that $\exp\bigl(|u|^{\frac{n}{n-1}}\bigr) \in L^p(\Omega)$ for any $p < \infty$, and there exists a limiting exponent $\alpha_n > 0$ such that the map

$$W_0^{1,n}(\Omega) \ni u \mapsto \exp\bigl(|u|^{\frac{n}{n-1}}\bigr) \in L^p(\Omega) \tag{4.16}$$

is bounded on the unit ball

$$B_1\bigl(0; W_0^{1,n}(\Omega)\bigr) = \left\{ u \in W_0^{1,n}(\Omega) \,;\, \int_\Omega |\nabla u|^n \, dx \le 1 \right\}$$

if and only if $p \le \alpha_n$.

Quite surprisingly, and in sharp contrast to the case $kp < n$ considered previously, Carleson-Chang [1] were able to establish that

$$\sup_{u \in B_1(0; W_0^{1,n}(\Omega))} \int_\Omega \exp\bigl(\alpha_n |u|^{\frac{n}{n-1}}\bigr) \, dx$$

is attained if Ω is a ball. Struwe [17] then showed that also for domains Ω that are close to a ball in measure the supremum is attained. Finally, Flucher [1] established the existence of an extremal function on any domain, if $n = 2$; see also Bandle-Flucher [1].

Moreover, at least in the case that Ω is a 2-ball (Struwe [17]), even for sufficiently small numbers $p > \alpha_2 = 4\pi$ the functional

$$E_p(u) = \int_\Omega \exp(p|u|^2) \, dx$$

admits a *relative* maximizer in $B_1\bigl(0; H_0^{1,2}(\Omega)\bigr)$. Thus, and since $E_p(u)$ by the result of Moser is unbounded on $B_1\bigl(0; H_0^{1,2}(\Omega)\bigr)$, we are led to expect the existence of a further critical point of saddle type, for any $p > 4\pi$ sufficiently close to 4π. In Struwe [17] such "unstable" critical points were, in fact, shown to exist for almost all such numbers $p > 4\pi$ by minimax methods as we shall describe in Chapter II. Note that for $p > 4\pi$ we are dealing with a "supercritical" variational problem and quite delicate techniques are needed to overcome the possible "loss of compactness". It is an open problem whether a similar result holds for *all* $p > 4\pi$ sufficiently close to 4π and for any domain; moreover, the extension of this result to higher dimensions is open.

(2°) Problems with exponential nonlinearities related to the embedding (4.16) above are studied, for instance, by Adimurthi-Yadava [1]. The embedding (4.16) is also relevant for the study of nonlinearities like

$$g(u) = e^{2u} \ ,$$

arising in the 2-dimensional Kazdan-Warner problem (see Moser [5], Chang-Yang [1]) or in the uniformization problem, that is, the Yamabe problem for surfaces; compare Section III.4.

5. Ekeland's Variational Principle

In general it is not clear that a bounded and lower semi-continuous functional E actually attains its infimum. The analytic function $f(x) = \arctan x$, for example, neither attains its infimum nor its supremum on the real line.

A variant due to Ekeland [1] of Dirichlet's principle, however, permits one to construct minimizing sequences for such functionals E whose elements u_m each minimize a functional E_m, for a sequence of functionals E_m converging locally uniformly to E.

5.1 Theorem. *Let M be a complete metric space with metric d, and let $E: M \to \mathbb{R} \cup +\infty$ be lower semi-continuous, bounded from below, and $\not\equiv \infty$.*
Then for any $\varepsilon, \delta > 0$, any $u \in M$ with

$$E(u) \leq \inf_M E + \varepsilon,$$

there is an element $v \in M$ strictly minimizing the functional

$$E_v(w) \equiv E(w) + \frac{\varepsilon}{\delta} d(v, w) \ .$$

Moreover, we have

$$E(v) \leq E(u), \quad d(u, v) \leq \delta \ .$$

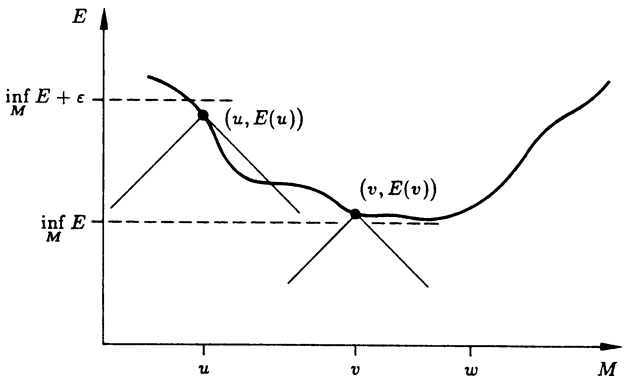

Fig. 5.1. Comparing E with E_v. v is a strict absolute minimizer of E_v if and only if the downward cone of slope ε/δ with vertex at $\big(v, E(v)\big)$ lies entirely below the graph of E

Proof. Denote $\alpha = \frac{\varepsilon}{\delta}$ and define a partial ordering on $M \times \mathbb{R}$ by letting

$$(v, \beta) \leq (v', \beta') \Leftrightarrow (\beta' - \beta) + \alpha\, d(v, v') \leq 0 \ . \tag{5.1}$$

This relation is easily seen to be reflexive, identitive, and transitive:

$$\begin{aligned}
&(v, \beta) \leq (v, \beta) \ , \\
&(v, \beta) \leq (v', \beta') \wedge (v', \beta') \leq (v, \beta) \Leftrightarrow v = v', \ \beta = \beta', \\
&(v, \beta) \leq (v', \beta') \wedge (v', \beta') \leq (v'', \beta'') \Rightarrow (v, \beta) \leq (v'', \beta'') \ .
\end{aligned}$$

Moreover, if we denote

$$S = \{(v, \beta) \in M \times \mathbb{R} \ ; \ E(v) \leq \beta\} \ ,$$

by lower semi-continuity of E, S is closed in $M \times \mathbb{R}$. To complete the proof we need a lemma.

5.2 Lemma. *S contains a maximal element (v, β) with respect to the partial ordering $\leq$ on $M \times \mathbb{R}$ such that $(u, E(u)) \leq (v, \beta)$.*

Proof. Let $(v_1, \beta_1) = (u, E(u))$ and define a sequence (v_m, β_m) inductively as follows: Given (v_m, β_m), define

$$\begin{aligned}
S_m &= \{(v, \beta) \in S \ ; \ (v_m, \beta_m) \leq (v, \beta)\} \\
\mu_m &= \inf\{\beta \ ; \ (v, \beta) \in S_m\} \\
&\geq \inf\{E(v) \ ; \ (v, \beta) \in S_m\} \geq \inf_M E =: \mu_0 \ .
\end{aligned}$$

Note that $\mu_m \leq \beta_m$; moreover, $S_m = \{(v_m, \beta_m)\}$, if $\mu_m = \beta_m$. Now let $(v_{m+1}, \beta_{m+1}) \in S_m$ be chosen such that

$$\beta_m - \beta_{m+1} \geq \frac{1}{2}(\beta_m - \mu_m) \ . \tag{5.2}$$

Note that by transitivity of $\leq$ the sequence S_m is nested: $S_1 \supset S_2 \supset \ldots \supset S_m \supset S_{m+1} \supset \ldots$. Hence also $\ldots \leq \mu_m \leq \mu_{m+1} \leq \ldots \leq \beta_{m+1} \leq \beta_m \leq \ldots$. By induction, from (5.2) we obtain

$$\begin{aligned}
\beta_{m+1} - \mu_{m+1} &\leq \beta_{m+1} - \mu_m \\
&\leq \frac{1}{2}(\beta_m - \mu_m) \leq \ldots \leq \left(\frac{1}{2}\right)^m (\beta_1 - \mu_1) \ .
\end{aligned}$$

Therefore, by definition of S_m, for any $m \in \mathbb{N}$ and any $(v, \beta) \in S_m$ we have

$$\begin{aligned}
|\beta_m - \beta| = \beta_m - \beta \leq \beta_m - \mu_m \leq C \left(\frac{1}{2}\right)^m \ , \\
d(v_m, v) \leq \alpha^{-1}(\beta_m - \beta) \leq C\alpha^{-1} \left(\frac{1}{2}\right)^m \ .
\end{aligned} \tag{5.3}$$

In particular, $\big((v_m, \beta_m)\big)_{m\in\mathbb{N}}$ is a Cauchy sequence in $M \times \mathbb{R}$. Thus, by completeness of M, $\big((v_m, \beta_m)\big)$ converges to some limit $(\overline{v}, \overline{\beta}) \in \bigcap_{m\in\mathbb{N}} S_m$. By transitivity, clearly $(u, E(u)) = (v_1, \beta_1) \leq (\overline{v}, \overline{\beta})$. Moreover, $(\overline{v}, \overline{\beta})$ is maximal. In fact, if $(\overline{v}, \overline{\beta}) \leq (\tilde{v}, \tilde{\beta})$ for some $(\tilde{v}, \tilde{\beta}) \in M \times \mathbb{R}$, then also $(v_m, \beta_m) \leq (\tilde{v}, \tilde{\beta})$ for all m, and $(\tilde{v}, \tilde{\beta}) \in S_m$ for all m. Letting $(v, \beta) = (\tilde{v}, \tilde{\beta})$ in (5.3) we infer that $(v_m, \beta_m) \to (\tilde{v}, \tilde{\beta})$ $(m \to \infty)$, whence $(\tilde{v}, \tilde{\beta}) = (\overline{v}, \overline{\beta})$, as desired. □

Proof of Theorem 5.1 (Completed). Let (v, β) be maximal in S with $(u, E(u)) \leq (v, \beta)$. Comparing with $(v, E(v)) \in S$ at once yields $\beta = E(v)$. By definition (5.1) the statement $(u, E(u)) \leq (v, E(v))$ translates into the estimate

$$E(v) - E(u) + \alpha\, d(u, v) \leq 0\ ;$$

in particular this implies

$$E(v) \leq E(u)$$

and

$$d(u, v) \leq \alpha^{-1}\,(E(u) - E(v)) \leq \frac{\delta}{\varepsilon}\left(\inf_M E + \varepsilon - \inf_M E\right) = \delta\ .$$

Finally, if $w \in M$ satisfies

$$E_v(w) = E(w) + \alpha\, d(v, w) \leq E(v) = E_v(v)\ ,$$

by Definition 5.1 we have $(v, E(v)) \leq (w, E(w))$. Hence $w = v$ by maximality of $(v, E(v))$; that is, v is a strict minimizer of E_v, as claimed. □

5.3 Corollary. *If V is a Banach space and $E \in C^1(V)$ is bounded from below, there exists a minimizing sequence (v_m) for E in V such that*

$$E(v_m) \to \inf_V E, \quad DE(v_m) \to 0 \qquad \text{in } V^* \qquad \text{as } m \to \infty\ .$$

Proof. Choose a sequence (ε_m) of numbers $\varepsilon_m > 0$, $\varepsilon_m \to 0$ $(m \to \infty)$. For $m \in \mathbb{N}$ choose $u_m \in V$ such that

$$E(u_m) \leq \inf_V E + \varepsilon_m^2\ .$$

For $\varepsilon = \varepsilon_m^2$, $\delta = \varepsilon_m$, $u = u_m$ determine an element $v_m = v$ according to Theorem 5.1, satisfying

$$E(v_m) \leq E(v_m + w) + \varepsilon_m\, \|w\|_V$$

for all $w \in V$. Hence

$$\|DE(v_m)\|_{V^*} = \limsup_{0\neq\|w\|_V\to 0} \frac{E(v_m) - E(v_m + w)}{\|w\|_V} \leq \varepsilon_m \to 0\ ,$$

as claimed. □

In Chapter II we will re-encounter the special minimizing sequences of Corollary 5.3 as "Palais-Smale sequences". Compactness of such sequences by Corollary 5.3 turns out to be a sufficient condition for the existence of a minimizer for a differentiable functional E which is bounded from below on a Banach space V. Moreover, we shall see that the compactness of Palais-Smale sequences (under suitable assumptions on the topology of the level sets of E) will also guarantee the existence of critical points of saddle type.

However, before turning our attention to critical points of general type we sketch another application of Ekeland's variational principle.

Existence of Minimizers for Quasi-convex Functionals

Theorem 5.1 may be used to construct minimizing sequences for variational integrals enjoying better smoothness properties than can a-priori be expected. We present an example due to Marcellini-Sbordone [1].

5.4 Example. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN} \to \mathbb{R}$ be a Carathéodory function satisfying the growth and coercivity conditions

$$|f(x,u,p)| \le C\big(1+|u|^s+|p|^s\big) \text{ for some } s>1\ , C\in\mathbb{R} \tag{5.4}$$

and

$$f(x,u,p) \ge |p|^s. \tag{5.5}$$

Moreover, suppose f is quasi-convex in the sense of Morrey [3]; that is, for almost every $x_0\in\Omega, u_0\in\mathbb{R}^N, p_o\in\mathbb{R}^{nN}$, and any $\varphi\in H_0^{1,s}(\Omega)$ there holds

$$\frac{1}{\mathcal{L}^n(\Omega)}\int_\Omega f\big(x_0,u_0,p_0+D\varphi(x)\big)\,dx \ge f(x_0,u_0,p_0). \tag{5.6}$$

Note that by Jensen's inequality condition (5.6) is weaker than requiring f to be convex in p. For $u\in H^{1,s}(\Omega;\mathbb{R}^N)$ now set

$$E(u)=E(u;\Omega)=\int_\Omega f\big(x,u(x),\nabla u(x)\big)\,dx\ .$$

By a result of Fusco [1], conditions (5.4) and (5.6) alone already suffice to ensure weak lower semi-continuity of E in $H^{1,t}(\Omega;\mathbb{R}^N)$ for any $t>s$; on the other hand, an example by Murat and Tartar shows that this is no longer true for $t=s$; see for instance Marcellini-Sbordone [1; Section 2].

But if, in addition to (5.4) and (5.6), we also assume (5.5), Marcellini-Sbordone [1] succeed in finding a minimizing sequence for E for given boundary data u_0 which is locally bounded in $H^{1,t}$ for some $t>s$ and therefore weakly accumulates at a minimizer of E. Their proof, which we shall presently explain, is based on Ekeland's variational principle and Giaquinta-Modica's [1] adaption of a result of Gehring [1], Lemma 5.6 below.

Finally, it was shown by Acerbi-Fusco [1] that E, in fact, for any $s \geq 1$ is weakly lower semi-continuous in $H^{1,s}(\Omega;\mathbb{R}^N)$ whenever f only satisfies (5.4), (5.6) and the condition $f(x,u,p) \geq 0$.

Regularity results for minimizers of (strictly) quasi-convex functionals have been obtained by Evans [1], Evans-Gariepy [1], and Giaquinta-Modica [2].

5.5 Theorem. *Under the above hypotheses (5.4)–(5.6) on f, for any $u_0 \in H^{1,s}(\Omega;\mathbb{R}^N)$ there is a minimizing sequence (u_m) for E on $\{u_0\}+H_0^{1,s}(\Omega;\mathbb{R}^N)$ which is locally bounded in $H^{1,t}$ for some $t > s$.*

Proof. Choose

$$M = \{u_0\} + H_0^{1,1}(\Omega;\mathbb{R}^N)$$

with metric d derived from the $H_0^{1,1}$-norm

$$d(u,v) = \int_\Omega |\nabla u - \nabla v|\, dx\ .$$

Note that by Fatou's lemma $E\colon M \to \mathbb{R} \cup +\infty$ is lower semi-continuous with respect to d. Let $u_m \in \{u_0\} + H_0^{1,s}(\Omega;\mathbb{R}^N) \subset M$ be a minimizing sequence. By Theorem 5.1, if we let

$$\varepsilon_m^2 = E(u_m) - \inf\{E(u)\ ;\ u \in M\}\ ,$$

$\delta_m = \varepsilon_m$, we can choose a new minimizing sequence (v_m) in M such that each v_m minimizes the functional

$$E_m(w) = E(w) + \varepsilon_m\, d(v_m, w)\ .$$

In particular, for each $\Omega' \subset\subset \Omega$ and any $w \in \{v_m\} + H_0^{1,s}(\Omega';\mathbb{R}^N)$ there holds

$$E(v_m;\Omega') \leq E(w;\Omega') + \varepsilon_m \int_{\Omega'} |\nabla w - \nabla v_m|\, dx\ . \tag{5.7}$$

Choose $x_0 \in \Omega$ and for $R < \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$, $N \in \mathbb{N}$, $1 \leq \nu \leq N$, choose $\varphi = \varphi_\nu \in C_0^\infty\big(B_{(1+\nu/N)R}(x_0)\big) \subset C_0^\infty\big(B_{2R}(x_0)\big)$ satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{(1+(\nu-1)/N)R}(x_0)$, $|\nabla\varphi| \leq CN/R$ with C independent of N and R. Let

$$\overline{v}_m = \frac{1}{\mathcal{L}^n(B_{2R} \setminus B_R(x_0))} \int_{B_{2R}\setminus B_R(x_0)} v_m\, dx$$

denote the mean value of u over the annulus $B_{2R} \setminus B_R(x_0)$. Define

$$w = w_{m,\nu} = (1-\varphi)v_m + \varphi\overline{v}_m\ .$$

Then $w = v_m$ outside $B_{(1+\nu/N)R}(x_0)$ and by (5.5) and (5.7) we have

$$\begin{aligned}\int_{B_R(x_0)} |\nabla v_m|^s\, dx &\leq E(v_m; B_{(1+\nu/N)R}(x_0)) \leq E\big(w; B_{(1+\nu/N)R}(x_0)\big) \\ &+ \varepsilon_m\Big(\int_{B_{2R}(x_0)} \varphi|\nabla v_m|\, dx + \int_{B_{2R}\setminus B_R(x_0)} |\nabla\varphi|\, |v_m - \overline{v}_m|\, dx\Big)\ .\end{aligned} \tag{5.8}$$

By Poincaré's inequality

$$\int_{B_{2R}\setminus B_R(x_0)} |v_m - \overline{v}_m|\, dx \le cR \int_{B_{2R}\setminus B_R(x_0)} |\nabla v_m|\, dx;$$

see Theorem A.10 of Appendix A. Hence, in particular, the last term in the previous inequality is bounded by $CN\varepsilon_m \int_{B_{2R}(x_0)} |\nabla v_m|\, dx$.

By choice of w and condition (5.4), moreover, we may estimate

$$\begin{aligned} E\big(w; B_{(1+\nu/N)R}(x_0)\big) &\le C \int_{B_{(1+\nu/N)R}(x_0)} \big(1 + |w|^s + |\nabla w|^s\big)\, dx \\ &\le C \int_{B_{(1+\nu/N)R}(x_0)} \big(1 + |v_m|^s + \varphi^s |v_m - \overline{v}_m|^s\big)\, dx \\ &\quad + C \int_{B_{(1+\nu/N)R}(x_0)} \big((1-\varphi)^s |\nabla v_m|^s + |\nabla \varphi|^s |v_m - \overline{v}_m|^s\big)\, dx \\ &\le C \int_{B_{2R}(x_0)} \big(1 + |v_m|^s + |\nabla \varphi|^s |v_m - \overline{v}_m|^s\big)\, dx \\ &\quad + C \int_{B_{(1+\nu/N)R}\setminus B_{(1+(\nu-1)/N)R}(x_0)} |\nabla v_m|^s\, dx\ . \end{aligned} \tag{5.9}$$

Note that we also used Hölder's inequality to estimate

$$\int_{B_{2R}(x_0)} |\overline{v}_m|^s\, dx \le CR^{n(1-s)} \left(\int_{B_{2R}(x_0)} v_m\, dx \right)^s \le C \int_{B_{2R}(x_0)} |v_m|^s\, dx\ .$$

For $u \in \mathcal{L}^1_{loc}(\Omega)$, $B_R = B_R(x_0) \subset \Omega$ let

$$\fint_{B_R} u\, dx = \frac{1}{\mathcal{L}^n(B_R)} \int_{B_R} u\, dx$$

denote the mean value of u over B_R, etc. By the Poincaré-Sobolev inequality, interpolation, and Young's inequality, for any $\delta > 0$ we can bound

$$\begin{aligned} \fint_{B_{2R}(x_0)} |\nabla \varphi|^s |v_m - \overline{v}_m|^s\, dx &\le CN^s R^{-s} \fint_{B_{2R}\setminus B_R(x_0)} |v_m - \overline{v}_m|^s\, dx \\ &\le \delta \fint_{B_{2R}(x_0)} |\nabla v_m|^s\, dx + C(\delta, N) \left(\fint_{B_{2R}(x_0)} |\nabla v_m|\, dx \right)^s . \end{aligned}$$

Summing the estimates (5.8) over $1 \le \nu \le N$, we fill the annulus $B_{2R} \setminus B_R(x_0)$ on the right of (5.9), and with a uniform constant C_0 we obtain

$$\begin{aligned} N \fint_{B_R(x_0)} |\nabla v_m|^s\, dx &\le C(N) \fint_{B_{2R}(x_0)} \big(1 + |v_m|^s + |\nabla v_m|\big)\, dx \\ &\quad + C(\delta, N) \left(\fint_{B_{2R}(x_0)} |\nabla v_m|\, dx \right)^s + C_0(1 + \delta N) \fint_{B_{2R}} |\nabla v_m|^s\, dx\ . \end{aligned}$$

Choosing $N \geq 2C_0 + 1$, $\delta = N^{-1} > 0$, upon dividing by N we find that

$$\fint_{B_R(x_0)} |\nabla v_m|^s \, dx \leq C \fint_{B_{2R}(x_0)} \left(1 + |v_m|^s + |\nabla v_m|\right) dx + C \left(\fint_{B_{2R}(x_0)} |\nabla v_m| \, dx \right)^s + \theta \fint_{B_{2R}(x_0)} |\nabla v_m|^s \, dx$$

with constants C and $\theta = \frac{2C_0}{2C_0+1} < 1$ independent of x_0, R, and m. (This idea in a related context first appears in Widman [1], Hildebrandt-Widman [1].)

Letting $g = |\nabla v_m|$, $h = \left(1 + |v_m|^s + |\nabla v_m|\right)^{1/s}$, $p = s \cdot \min\left\{\frac{n}{n-s}, s\right\} > s$ and observing that by Sobolev's embedding theorem $|v_m|^s \in L^p$, Theorem 5.5 now follows from the next lemma, due to Giaquinta-Modica [1].

5.6 Lemma. *Suppose Ω is a domain in $\mathbb{R}^n$ and $0 \leq g \in L^s(\Omega)$, $h \in L^p(\Omega)$ for some $p > s > 1$. Assume that for any $x_0 \in \Omega$ and $0 < R < \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$ there holds*

$$\fint_{B_R(x_0)} g^s \, dx \leq b \fint_{B_{2R}(x_0)} h^s \, dx + c \left(\fint_{B_{2R}(x_0)} g \, dx \right)^s + \theta \fint_{B_{2R}(x_0)} g^s \, dx$$

with uniform constants $\theta < 1$, b, and c independent of x_0 and R. Then there exists $\varepsilon > 0$, depending only on b, c, θ, p, s, and n, such that $g \in L^t_{loc}(\Omega)$ for $s < t < s + \varepsilon \leq p$, and for any $x_0 \in \Omega$, $0 < R < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$, and any such t there holds

$$\left(\fint_{B_R(x_0)} g^t \, dx \right)^{\frac{1}{t}} \leq C \left\{ \left(\fint_{B_R(x_0)} h^t \, dx \right)^{\frac{1}{t}} + \left(\fint_{B_{2R}(x_0)} g^s \, dx \right)^{\frac{1}{s}} \right\},$$

with C possibly depending also on t.

The proof of Lemma 5.6 goes beyond the scope of this book; a reference is Giaquinta [1; Proposition V.1.1, p. 122].

5.7 Note. Besides various other applications, Ekeland's variational principle has given rise to new interpretations of the mountain-pass lemma and its variants that we discuss in Section II.6; see, for instance, De Figueiredo [1] or Mawhin-Willem [1; Chapter 4.1] for an exposition.

The idea of choosing special minimizing sequences to ensure convergence towards a minimizer already appears in the work of Hilbert [1] and Lebesgue [1]. In their solution of Dirichlet's problem they use barriers and the "Courant-Lebesgue lemma", that also inspired our proof of Lemma 2.8 above, to ensure the equicontinuity and hence compactness of a suitably constructed minimizing sequence for Dirichlet's integral. (The compactness criterion for families of continuous functions on a compact domain was known from an earlier – though unsuccessful – attempt at solving Dirichlet's problem by Arzéla [1] in 1897.)

6. Duality

Let V be a Banach space and suppose $G: V \to \mathbb{R} \cup +\infty$ is lower semi-continuous and convex. Geometrically, convexity of G is equivalent to convexity of the epigraph of G

$$\operatorname{epi}(G) = \left\{(v, \beta) \in V \times \mathbb{R} \ ; \ \beta \geq G(v)\right\} ,$$

while lower semi-continuity is equivalent to the closedness of $\operatorname{epi}(G)$. By the Hahn-Banach separation theorem any closed convex set can be represented as the intersection of the closed half-spaces which contain it, bounded by support hyperplanes. Hence, for any lower semi-continuous, convex $G: V \to \mathbb{R} \cup +\infty$ there exists a set L_G of continuous affine maps such that for any $v \in V$ there holds

$$G(v) = \sup \{l(v) \ ; \ l \in L_G\} ,$$

see, for instance, Ekeland-Temam [1; Proposition I.3.1]. Moreover, at any point $v \in V$ where G is locally bounded there is a "support function" $l_v \in L_G$ such that $l_v(v) = G(v)$. The set of slopes of support functions

$$\partial G(v) = \{Dl \ ; \ l \in L_G \ , \ l(v) = G(v)\}$$

is called the subdifferential of G at v.

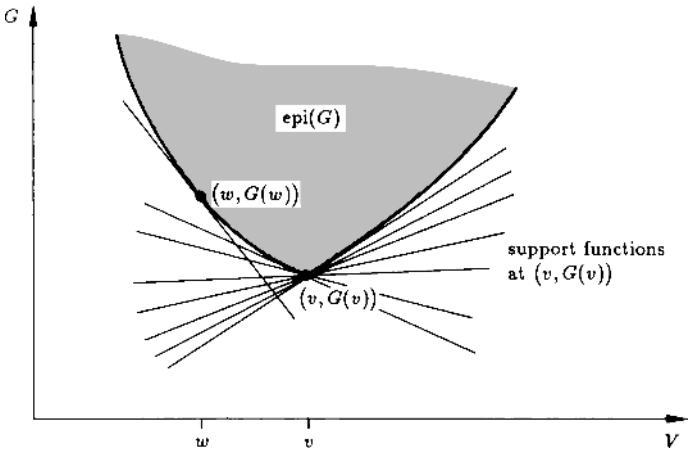

Fig. 6.1.

6.1 Lemma. *Suppose $G: V \to \mathbb{R} \cup +\infty$ is lower semi-continuous and convex. If G is (Gâteaux) differentiable at v, then $\partial G(v) = \{DG(v)\}$. Conversely, if G is locally bounded near v and if $\partial G(v) = \{v^*\}$ is single-valued, then G is Gâteaux-differentiable at v with $D_w G(v) = \langle w, v^* \rangle$ for all $w \in V$.*

(See, for instance, Ekeland-Temam [1; Proposition I.5.3] for a proof.)

6.2 The Legendre-Fenchel transform. For a function $G: V \to \mathbb{R} \cup +\infty$, $G \not\equiv +\infty$, not necessarily convex, the function $G^*: V^* \to \mathbb{R} \cup +\infty$, given by

$$G^*(v^*) = \sup \{\langle v, v^* \rangle - G(v) \ ; \ v \in V\} \ ,$$

defines the Legendre-Fenchel transform of G.

Note that G^* is the pointwise supremum of affine maps, hence G^* is lower semi-continous and convex; moreover, the affine function l_v^*, given by

$$l^*(v^*) = \langle v, v^* \rangle - G(v) \ ,$$

is a support function of G^* at v^* if and only if

$$G^*(v^*) + G(v) = \langle v, v^* \rangle \ . \tag{6.1}$$

For lower semi-continuous convex functions G the situation is, in fact, symmetric in G and G^*. To see this, introduce $G^{**} = (G^*)^*\big|_V$. Note that the Legendre-Fenchel transform reverses order, that is, $\tilde{G} \leq G$ implies $\tilde{G}^* \geq G^*$. Hence, applying the transform twice preserves order: $\tilde{G} \leq G$ implies $\tilde{G}^{**} \leq G^{**}$. Moreover, for affine functions $l^{**} = l$. Thus, for any affine map $l \leq G$ we have

$$l(v) = l^{**}(v) \leq G^{**}(v) = \sup_{v^* \in V^*} \left\{ \langle v, v^* \rangle - \sup_{w \in V} \left\{ \langle w, v^* \rangle - G(w) \right\} \right\} \leq G(v) \ .$$

(Choose $w = v$ inside $\{\ldots\}$.)

It follows that G^{**} is the largest lower semi-continuous convex function below G, and $G^{**} = G$ if and only if G is lower semi-continuous and convex. Our previous discussion implies:

6.3 Lemma. *Suppose $G: V \to \mathbb{R} \cup +\infty$, $G \not\equiv +\infty$, is lower semi-continuous and convex, and let G^* be its Legendre-Fenchel transform. Then (6.1) is equivalent to either one of the relations $v \in \partial G^*(v^*)$ or $v^* \in \partial G(v)$.*

Proof. In fact, if $v \in \partial G^*(v^*)$, there is $\beta \in \mathbb{R}$ such that l^*, given by

$$l^*(w^*) = \langle v, w^* \rangle - \beta,$$

belongs to L_{G^*} and satisfies $l^*(v^*) = G^*(v^*)$.

Since, by definition,

$$G^*(v^*) \geq \langle v, v^* \rangle - G(v),$$

we conclude that $G(v) \geq \beta$. On the other hand, since $G = G^{**} \leq (l^*)^*$, we have

$$G(v) \leq \sup_{w^*} \left\{ \langle v, w^* \rangle - l^*(w^*) \right\} = \beta,$$

and it follows that $G(v) = \beta$; that is, we obtain (6.1). The same reasoning shows that the assumption $v^* \in \partial G(v)$ implies (6.1). The converse is immediate from the definitions of G^* and $G = G^{**}$. ∎

In particular, if $G \in C^1(V)$ is strictly convex, which implies that

$$\langle v - w, DG(v) - DG(w) \rangle > 0 \qquad \text{if } v \neq w \ ,$$

then DG is injective, G^* is finite on the range of DG, and $\partial G^*\big(DG(v)\big) = \{v\}$ for any $v \in V$.

If, in addition, DG is strongly monotone and coercive in the sense that for all $v, w \in V$ there holds

$$\langle v - w, DG(v) - DG(w) \rangle \geq \alpha\big(\|v - w\|\big)\|v - w\|$$

with a non-decreasing function $\alpha\colon [0, \infty[\to [0, \infty[$ vanishing only at 0 and such that $\alpha(r) \to \infty$ as $r \to \infty$, then $DG\colon V \to V^*$ is also surjective; see, for instance, Brezis [1; Corollary 2.4, p. 31]. Moreover, ∂G^* – and hence G^* – are locally bounded near any $v^* \in V^*$. Thus, by Lemma 6.1, G^* is Gâteaux differentiable with $DG^*(v^*) = v$ for any $v^* = DG(v)$. Finally, from the estimate

$$\|v^* - w^*\| \geq \frac{\langle DG^*(v^*) - DG^*(w^*), v^* - w^* \rangle}{\|DG^*(v^*) - DG^*(w^*)\|} \geq \alpha\big(\|DG^*(v^*) - DG^*(w^*)\|\big)$$

for all $v^*, w^* \in V^*$ it follows that DG^* is continuous; that is, $G^* \in C^1(V^*)$. We conclude that for any strictly convex function $G \in C^1(V)$ such that DG is strongly monotone, the differential DG is a homeomorphism of V onto its dual V^*. (In the following we apply these results only in a finite-dimensional setting. A convenient reference in this case is Rockafellar [1; Theorem 26.5, p. 258].)

Hamiltonian Systems

We now apply these concepts to the solution of Hamiltonian systems. For a Hamiltonian $H \in C^2(\mathbb{R}^{2n})$ and the standard symplectic form $\mathcal{J}$ on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$,

$$\mathcal{J} = \begin{pmatrix} 0 & -id \\ id & 0 \end{pmatrix} \ ,$$

where id is the identity map on $\mathbb{R}^n$, we consider the ordinary differential equation

$$\dot{x} = \mathcal{J}\,\nabla H(x) \ . \tag{6.2}$$

Note that by anti-symmetry of $\mathcal{J}$ we have

$$\frac{d}{dt} H\big(x(t)\big) = \nabla H(x) \cdot \dot{x}(t) = 0 \ ,$$

that is, $H\big(x(t)\big) \equiv const.$ along any solution of (6.2) and any "energy-surface" $H = const.$ is invariant under the flow (6.2).

6.4 Periodic solutions. One would like to understand the global structure of the set of trajectories of (6.2) and their asymptotic behavior. This is motivated of course by celestial mechanics, where questions of "stable and ramdon motion" (see Moser [4]) also seem to be of practical importance. However, with exception of the – very particular – "completely integrable" case, this program is far too complex to be dealt with as a whole. Therefore, one is interested in sub-systems of the flow (6.2) such as stationary points, periodic orbits, invariant tori, or quasi-periodic solutions. While stationary points in general do not reveal too much about the system, it turns out that – C^1-generically at least – periodic orbits of (6.2) are dense on a compact energy surface $H = const.$; see Pugh-Robinson [1]. Such a result seems to have already been envisioned by Poincaré [1; Tome 1, Article 36]. For particular systems, however, such results are much harder to obtain. In fact, the question whether any *given* energy surface carries a periodic solution of (6.2) has only recently been answered; see Section II.9 for more details.

In this section we consider the special class of convex Hamiltonians. This class includes as a simple model case the harmonic oscillator, described by the Hamiltonian

$$H(p,q) = \frac{1}{2}\big(|p|^2 + |q|^2\big), x = (p,q) \in \mathbb{R}^2 \ ,$$

for which (6.2) possesses periodic solutions x of any given energy $H(x(t)) = \beta > 0$, all having the same period 2π. In the general case, the following result of Rabinowitz [5] and Weinstein [2], extending earlier work of Seifert [1], holds.

6.5 Theorem. *Suppose $H \in C^1(\mathbb{R}^{2n})$ is strictly convex, non-negative and coercive with $H(0) = 0$. Then for any $\alpha > 0$ there is a periodic solution $x \in C^1\big(\mathbb{R}\ ;\ \mathbb{R}^{2n}\big)$ of (6.2) with $H(x(t)) = \alpha$ for all t. The period T is not specified.*

Remarks. Seifert and Weinstein essentially approached problem (6.2) using differential geometric methods, that is, by interpreting solutions of (6.2) as geodesics in a suitable Riemannian or Finsler metric (the so-called Jacobi metric). Rabinowitz' proof of Theorem 6.5 revolutionized the study of Hamiltonian systems in the large as it introduced variational methods to this field.

Note that (6.2) may be interpreted as the Euler-Lagrange equation of the functional

$$E(x) = \frac{1}{2}\int_0^1 \langle x, \mathcal{J}\dot{x}\rangle \, dt \tag{6.3}$$

on the class

$$C_\alpha = \{x \in C^1(\mathbb{R};\mathbb{R}^{2n})\ ;\ x(t+1) = x(t),\ \int_0^1 H(x(t))\, dt = \alpha\}.$$

Indeed, at a critical point $x \in C_\alpha$ of E, by the Lagrange multiplier rule there exists $T \neq 0$ such that

$$\dot{x} = T\mathcal{J}\nabla H(x) \ ,$$

and scaling time by a factor T we obtain a T-periodic solution of (6.2) on the energy surface $H = \alpha$.

However, the integral (6.3) is not bounded from above or below. In fact, E is a quadratic form given by an operator $x \mapsto \mathcal{J}\dot{x}$ with *infinitely* many positive *and* negative eigenvalues. Owing to this complication, actually, for a long time it was considered hopeless to approach the existence problem for periodic solutions of (6.2) via the functional (6.3). Surprisingly, by methods that will be presented in Chapter II below, and by using a delicate approximation procedure, Rabinowitz was able to overcome these difficulties. In fact, his result is somewhat more general than stated above as it applies to compact, strictly star-shaped energy hypersurfaces.

On a compact, convex energy hypersurface – as was first observed by Clarke [2], [4] – by duality methods his original proof can be considerably simplified and the problem of finding a periodic solution of (6.2) can be recast in a way such that a solution again may be sought as a minimizer of a suitable "dual" variational problem. See also Clarke-Ekeland [1]. This is the proof we now present.

Later we shall study the existence of periodic solutions of Hamiltonian systems under much more general hypotheses; see Section II.8. Moreover, we shall study the existence of multiple periodic orbits; see Section II.5.

Proof of Theorem 6.5. In a first step we reformulate the problem in a way such that duality methods can be applied.

Note that by strict convexity and coerciveness of H the level surface $S_\alpha = H^{-1}(\{\alpha\})$ bounds a strictly convex neighborhood of 0 in $\mathbb{R}^{2n}$. Thus, for any ξ in the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ there exists a unique number $r(\xi) > 0$ such that $x = r(\xi)\xi \in S_\alpha$. By the implicit function theorem $r \in C^1(S^{2n-1})$. Replace H by the function

$$\tilde{H}(x) = \begin{cases} \alpha r\big(x/|x|\big)^{-q}|x|^q, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

where q is a fixed number $1 < q < 2$.

Note that $\tilde{H} \in C^1(\mathbb{R}^{2n})$ and is homogeneous of degree q on half-rays from the origin. Moreover, if we let $\tilde{S}_\alpha = \{x \in \mathbb{R}^{2n} \ ; \ \tilde{H}(x) = \alpha\}$ we have $\tilde{S}_\alpha = S_\alpha$ and hence $\nabla\tilde{H}(x)$ is proportional to $\nabla H(x)$. In fact, since $\tilde{H}$ increases in direction $\nabla H(x)$ at any point $x \in X_\alpha$, there exists a function $\lambda > 0$ such that

$$\nabla H(x) = \lambda(x)\nabla\tilde{H}(x) \ , \quad \text{at any} \quad x \in S_\alpha \ .$$

A periodic solution $\tilde{x}$ on $\tilde{S}_\alpha$ to (6.2) for $\tilde{H}$ after a change of parameter thus will yield a periodic solution

$$x(t) = \tilde{x}\big(s(t)\big)$$

to (6.2) on S_α for the original function H, where s solves

$$\dot{s} = \lambda\big(\tilde{x}(s)\big) \ .$$

This, incidentally, also shows that whether or not a level surface $H = const.$ carries a periodic solution of (6.2) is a question concerning the *surface* and the symplectic structure $\mathcal{J}$ – not the particular Hamiltonian H.

Finally, $\tilde{H}$ is strictly convex. Indeed, consider any point $x = \rho\xi \in \mathbb{R}^{2n}\backslash\{0\}$ with $\tilde{H}(x) = \beta > 0$. Note that

$$\tilde{S}_\beta = \{x \in \mathbb{R}^{2n} \ ; \ \tilde{H}(x) = \beta\} = \left(\frac{\beta}{\alpha}\right)^{1/q} S_\alpha \ .$$

Thus, the hyperplane through $(x, \beta) \in \mathbb{R}^{2n+1}$, parallel to the hyperplane spanned by

$$T_x\tilde{S}_\beta \cong T_{r(\xi)\xi}S_\alpha \subset \mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n+1}$$

together with the vector

$$(x, x \cdot \nabla\tilde{H}(x)) = (x, q\tilde{H}(x)) = (x, q\beta) \in \mathbb{R}^{2n+1} \ ,$$

is a support hyperplane for $epi(\tilde{H})$ which touches the graph of $\tilde{H}$ precisely at (x, β). Similarly, since $\tilde{H} \geq 0 = \tilde{H}(0)$, the hyperplane $\mathbb{R}^{2n} \times \{0\}$ is a support hyperplane at $(0, 0)$, and $\tilde{H}$ is strictly convex.

Hence in the following we may assume that $H = \tilde{H}$.

Let H^* be the Legendre-Fenchel transform of H. Note that, since H is homogeneous on rays of degree $q > 1$, the function H^* is everywhere finite. Moreover, $H^*(0) = 0, \ H^* \geq 0$. Also note that for a function H on $\mathbb{R}^{2n}$ which is homogeneous of degree $q > 1$, strict convexity implies strong monotonicity of the gradient in the sense that, with a continuous function $a = a(\rho) \geq 0$, vanishing only at $\rho = 0$, there holds

$$\langle x - y, \nabla H(x) - \nabla H(y)\rangle \geq a\left(\frac{|x-y|}{|x|+|y|}\right)\big(|x| + |y|\big)^{q-1}|x - y|.$$

Hence, arguing as in the discussion following Lemma 6.3, we find that $H^* \in C^1(\mathbb{R}^{2n})$. Finally, letting $p = \frac{q}{q-1} > 2$ be the conjugate exponent of q, we have

$$\begin{aligned} \frac{H^*(y)}{|y|^p} &= \sup\left\{\left\langle \frac{x}{|y|^{p-1}}, \frac{y}{|y|}\right\rangle - \frac{H(x)}{|y|^p} \ ; \ x \in \mathbb{R}^{2n}\right\} \\ &= \sup\left\{\left\langle \frac{x}{|y|^{p-1}}, \frac{y}{|y|}\right\rangle - H\left(\frac{x}{|y|^{p-1}}\right) \ ; \ x \in \mathbb{R}^{2n}\right\} \\ &= H^*\left(\frac{y}{|y|}\right) \ ; \end{aligned} \tag{6.4}$$

that is, H^* is homogeneous on rays of degree $p > 2$.

(At this point we should remark that the components of the variable x above include both position and momentum variables. Thus, although these are certainly related, the conjugate H^* of H differs from the usual Legendre transform of H which customarily only involves the momentum variables.)

Introduce the space

$$L_0^p = \left\{ y \in L^p\big([0,1]; \mathbb{R}^{2n}\big) \ ; \ \int_0^1 y \, dt = 0 \right\} \ .$$

Now, if $x \in C^1\big([0,1]; \mathbb{R}^{2n}\big)$ is a 1-periodic solution of (6.2), the function $y = -\mathcal{J}\dot{x} \in L_0^p$ solves the system of equations

$$y = -\mathcal{J}\dot{x} \ , \tag{6.5}$$

$$y = \nabla H(x) \ . \tag{6.6}$$

Equation (6.5) can be inverted (up to an integration constant $x_0 \in \mathbb{R}^n$) by introducing the integral operator

$$K\colon L_0^p \to H^{1,p}\big([0,1]; \mathbb{R}^{2n}\big) \ , \qquad (Ky)(t) = \int_0^t \mathcal{J}y \, dt \ .$$

By Lemma 6.3 relation (6.6) is equivalent to the relation $x = \nabla H^*(y)$. That is, system (6.5), (6.6) is equivalent to the system

$$x = Ky + x_0 \tag{6.5$'$}$$

$$x = \nabla H^*(y) \tag{6.6$'$}$$

for some $x_0 \in \mathbb{R}^{2n}$. The latter can be summarized in the single equation

$$\int_0^1 \big(\nabla H^*(y) - Ky\big) \cdot \eta \, dt = 0 \ , \qquad \forall \eta \in L_0^p \ . \tag{6.7}$$

Indeed, if $y \in L_0^p$ satisfies (6.7), it follows that

$$\nabla H^*(y) - Ky = const. = x_0 \in \mathbb{R}^n \ .$$

Hence y solves (6.5$'$), (6.6$'$) for some $x \in H^{1,p}\big([0,1]; \mathbb{R}^{2n}\big)$. Transforming back to (6.5), (6.6), from (6.6) we see that $y \in H^{1,p} \hookrightarrow C^0$, and therefore $x \in C^1\big([0,1]; \mathbb{R}^{2n}\big)$ is a 1-periodic solution of (6.2). Thus, (6.2) and its weak "dual" form (6.7) are in fact equivalent.

Now we can conclude the proof of Theorem 6.5: We recognize (6.7) as the Euler-Lagrange equation of the functional E^* on L_0^p, given by

$$E^*(y) = \int_0^1 \left(H^*(y) - \frac{1}{2}\langle y, Ky \rangle \right) \, dt \ .$$

Note that by (6.4) the functional E^* is Fréchet-differentiable and coercive on L_0^p. Moreover, E^* is the sum of a continuous convex and a compact quadratic

term, hence weakly lower semi-continuous. Thus, by Theorem 1.2, a minimizer $y^* \in L_0^p$ of E^* exists, solving (6.7). By (6.4) the quadratic term $-\int_0^1 \langle y, Ky\rangle\, dt$ in E^* dominates near $y = 0$. Since, for instance, $\langle y, Ky\rangle > 0$ for $y(t) = ae^{2\pi Jt} \in L_0^p$ with $a \in \mathbb{R}^{2n} \setminus \{0\}$, we have $\inf E^* < 0$, and $y^* \neq 0$. By the above discussion there is a constant x_0 such that the function $x = Ky^* + x_0$ solves (6.2). Since $y^* \neq 0$, also x is non-constant; hence $H(x(t)) = \beta > 0$. But $H = \tilde{H}$ is homogeneous on rays. Thus a suitably rescaled multiple $\tilde{x}$ of x, $\tilde{x} = (\frac{\alpha}{\beta})^{1/q} x(T\cdot)$, will satisfy (6.2) with $H(\tilde{x}) = \alpha$, as desired. □

Periodic Solutions of Nonlinear Wave-Equations

As a second example we consider the problem of finding a non-constant, time-periodic solution $u = u(x,t),\ 0 \le x \le \pi,\ t \in \mathbb{R}$, of the problem

$$Au = u_{tt} - u_{xx} = -u|u|^{p-2} \qquad \text{in }]0,\pi[\times\mathbb{R} \tag{6.8}$$

$$u(0,\cdot) = u(\pi,\cdot) = 0 \tag{6.9}$$

$$u(\cdot, t+T) = u(\cdot, t) \qquad \text{for all } t \in \mathbb{R}\ , \tag{6.10}$$

where $p > 2$ and the period T are given.

The following result again is due to Rabinowitz [6].

6.6 Theorem. *Suppose $T/\pi \in \mathbb{Q}$; then there exists a non-constant T-periodic weak solution $u \in L^p\big([0,\pi]\times\mathbb{R}\big)$ of problem (6.8)–(6.10).*

Remark. For simplicity, we consider only the case $T = 2\pi$; the general case $T/\pi \in \mathbb{Q}$ can be handled in a similar way. The situation, however, changes completely if T is not a rational multiple of the spatial period, in this case, a rational multiple of π. Whether or not Theorem 6.7 holds true in this case is an open problem which seems to call for techniques totally different from those we are going to describe. (See Bobenko-Kuksin [1] or Pöschel [1] for some recent results in this regard.)

Proof. Problem (6.8)–(6.10) can be interpreted as the Euler-Lagrange equations associated with a constrained minimization problem for the functional

$$E(u) = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi} \big(|u_x|^2 - |u_t|^2\big)\, dx\, dt$$

on the space

$$H = \big\{u \in H^{1,2}_{loc}\big([0,\pi]\times\mathbb{R}\big)\ ;\ u \text{ satisfies (6.9), (6.10)}\big\}\ ,$$

endowed with the $H^{1,2}$-norm on $\Omega = [0,\pi]\times[0,2\pi]$, subject to the constraint

$$\|u\|_{L^p(\Omega)} = 1\ .$$

However, E is unbounded on this set. Moreover, the operator $A = \partial_t^2 - \partial_x^2$ related to the second variation of E has infinitely many positive and negative eigenvalues and also possesses an infinite-dimensional kernel. Therefore – as in the case of Hamiltonian systems considered above – the direct methods do not immediately apply.

In order to convert (6.8)–(6.10) into a problem that we can handle, we write (6.8) as a system

$$v = Au \tag{6.11}$$

$$-v = u|u|^{p-2} = \nabla G(u) \ , \tag{6.12}$$

where $G(u) = \frac{1}{p}|u|^p$. Since G is strictly convex, (6.12) may be inverted using the Legendre-Fenchel transform of G,

$$G^*(v) = \sup \{uv - \frac{1}{p}|u|^p \ ; \ u \in \mathbb{R}\} = \frac{1}{q}|v|^q \ ,$$

where $1 < q < 2$ is the exponent conjugate to p, satisfying $\frac{1}{p} + \frac{1}{q} = 1$. By Lemma 6.3 then, (6.12) is equivalent to the equation

$$u = \nabla G^*(-v) = -v|v|^{q-2} \ . \tag{6.13}$$

In order to invert (6.11) we need to collect some facts about the wave operator A. In our exposition we basically follow Brezis-Coron-Nirenberg [1]. The representation formula (6.14)–(6.16) is due to Lovicarová [1].

6.7 Estimates for the wave operator A. For $T = 2\pi$ the spectrum $\sigma(A)$ and kernel N of A, acting on functions in $L^1(\Omega)$ satisfying (6.9), (6.10), can be characterized as follows:

$$\sigma(A) = \{j^2 - k^2 \ ; \ j \in \mathbb{N}, \ k \in \mathbb{N}_0\} \ ,$$

$$N = \Big\{ p(t+x) - p(t-x) \ ; \ p \in L^1_{loc}(\mathbb{R}), \ p(s+2\pi) = p(s) \text{ for almost all } s, \int_0^{2\pi} p \, dx = 0 \Big\} \ .$$

The last condition appearing in the definition of N is a normalization condition. N is closed in $L^1(\Omega)$; moreover, given $f \in L^1(\Omega)$ such that $\int_\Omega f\varphi \, dx \, dt = 0$ for all $\varphi \in N \cap L^\infty(\Omega)$, there exists a unique function $u \in C(\overline{\Omega})$, satisfying (6.9), (6.10), such that $Au = f$ and $\int_\Omega u\varphi \, dx \, dt = 0$ for all $\varphi \in N$. In fact, u is given explicitly as follows:

$$u(x,t) = \psi(x,t) + \big(p(t+x) - p(t-x)\big) \ ,$$

where ψ is constructed from a 2π-periodic extension of f to $[0,\pi] \times \mathbb{R}$ by using the fundamental solution of the wave operator; that is,

$$\psi(x,t) = -\frac{1}{2}\int_x^\pi \left(\int_{t-(\xi-x)}^{t+(\xi-x)} f(\xi,\tau)\, d\tau \right) d\xi + c\frac{\pi - x}{\pi} \ , \tag{6.14}$$

with

$$c = \frac{1}{2}\int_0^\pi \left(\int_{t-\xi}^{t+\xi} f(\xi,\tau)\, d\tau \right) d\xi \ . \tag{6.15}$$

Note that c is constant; here, the fact that f is orthogonal to N is used. The choice of c now guarantees that u satisfies the boundary condition (6.9); moreover, periodicity of f implies (6.10). Finally, choosing

$$p(s) = \frac{1}{2\pi}\int_0^\pi \big[\psi(\xi, s+\xi) - \psi(\xi, s-\xi)\big]\, d\xi \tag{6.16}$$

ensures that u is L^2-orthogonal to N, as desired.

Formulas (6.14)–(6.16) determine an operator $K = A^{-1}$ from the weak orthogonal complement of N

$$N^\perp = \left\{ f \in L^1(\Omega) \ ; \ \int_\Omega f\,\varphi \, dx\, dt = 0 \qquad \text{for all } \varphi \in N \cap L^\infty(\Omega) \right\}$$

into $C(\overline{\Omega})$ satisfying the condition

$$\|Kf\|_{L^\infty} \le c\, \|f\|_{L^1} \ . \tag{6.17}$$

Moreover, for $f \in N^\perp \cap L^q(\Omega)$, $q > 1$, we have $Kf \in C^\alpha(\overline{\Omega})$, with $\alpha = 1 - 1/q > 0$ and

$$\|Kf\|_{C^\alpha} \le c\, \|f\|_{L^q} \ ; \tag{6.18}$$

in particular, K is a compact, selfadjoint linear operator of $N^\perp \cap L^2(\Omega)$ into itself, with eigenvalues $1/(j^2 - k^2)$, $j \in \mathbb{N}$, $k \in \mathbb{N}_0$, $j \ne k$. (6.17) and (6.18) are easy consequences of (6.14)–(6.16) and Hölder's inequality.

Proof of Theorem 6.6. Fix $q = \frac{p}{p-1}$ and let $V = N^\perp \cap L^q(\Omega)$, endowed with the L^q-norm. By (6.18) the operator $K\colon V \to L^p(\Omega)$ is compact. Define

$$E^*(v) = \frac{1}{2}\int_\Omega (Kv)\, v \, dx\, dt \ ;$$

clearly $E^* \in C^1(V)$. Moreover, since K is compact, it follows that E^* is weakly lower semi-continuous. Restrict E^* to the unit sphere

$$M = \{ v \in V \ ; \ \|v\|_{L^q} = 1 \}$$

in L^q and consider a minimizing sequence (v_m) for E^* in M. We may assume that $v_m \rightharpoonup v^*$ weakly in L^q, whence by weak lower semi-continuity

$$E^*(v^*) \le \liminf_{m\to\infty} E^*(v_m) = \inf \{ E^*(v) \ ; \ v \in M \} < 0 \ . \tag{6.19}$$

(To verify the last inequality recall that K possesses also negative eigenvalues.)

In particular, $v^* \neq 0$ and $v^*/\|v^*\|_{L^q} \in M$. But then, since $E^*(\rho v) = \rho^2 E^*(v)$ for all v, by (6.19) we must have $\|v^*\|_{L^q} = 1$, and $v^* \in M$ minimizes E^* on M.

By the Lagrange multiplier rule, v^* satisfies the equation

$$\int_\Omega \big(Kv^* + \mu v^*|v^*|^{q-2}\big)\varphi \, dx \, dt = 0 \ , \qquad \text{for all } \varphi \in V \ , \tag{6.20}$$

with a Lagrange parameter $\mu \in \mathbb{R}$. Choosing $\varphi = v^*$ in (6.20) we realize that $\mu = -2E^*(v^*) > 0$. Scaling v^* suitably, we obtain a non-constant function $v \in V$, satisfying (6.20) with $\mu = 1$. But then v satisfies

$$Kv + v|v|^{q-2} \in N \cap L^p \ .$$

Letting $u = -v|v|^{q-2} \in L^p$, thus there exists $\psi \in N \cap L^p$ such that

$$\begin{aligned} u &= Kv + \psi \ , \\ u &= -v|v|^{q-2} \ . \end{aligned}$$

But this system of equations is equivalent to (6.11), (6.12), and we conclude that u is a non-constant solution of the equation

$$Au + u|u|^{p-2} = 0 \ ,$$

satisfying the boundary and periodicity conditions (6.9), (6.10), as desired. □

6.8 Notes. (1°) Actually, the solution obtained above is of class L^∞, see Brezis-Coron-Nirenberg [1; p. 672 f.].

(2°) Theorem 6.6 remains valid for a large class of semilinear equations

$$u_{tt} - x_{xx} + g(u) = 0 \tag{6.21}$$

involving functions g sharing the qualitative behavior of a superlinear monomial $g(u) = u|u|^{p-2}$; see Rabinowitz [6], Brezis-Coron-Nirenberg [1]. It is not even in general necessary that g is monotone (Coron [1]) – although clearly the proof given above cannot be extended to such a case.

In the case g is smooth (C^∞) and strictly increasing, it was shown by Rabinowitz [6] and Brezis-Nirenberg [1] that any bounded solution u to (6.21), with boundary conditions (6.9), (6.10), is of class C^∞.
(3°) Further aspects of problem (6.8)–(6.10) have been studied by Salvatore [1] and Tanaka [1]. See Chapter II, Remark 7.3 and Notes 10.6 for references.

Other applications of duality methods to semilinear wave equations and related problems have been given by Willem [1], [2].

7. Minimization Problems Depending on Parameters

Quite often in the calculus of variations a functional to be minimized may depend on parameters. In particular, this is the case if we attempt to approximate a minimization problem with constraints by a family of unconstrained variational problems involving a "penalty term".

7.1 Penalty method. Suppose that V is a Banach space and let E and G be non-negative functionals on V. We seek to minimize E on the set

$$M = \{u \in V \ ; \ G(u) = 0\}$$

of admissible functions.

Without any smoothness and non-degeneracy assumptions, the set M may be very "rough", and a direct approach to the constrained minimization problem may be rather cumbersome. Thus, instead of tackling this constrained optimization problem directly, it is often more convenient to find approximate solutions by minimizing, for $\epsilon > 0$, the functional

$$E_\epsilon(u) = E(u) + \epsilon^{-1} G(u), \qquad u \in V.$$

That is, we relax the constraint $G(u) = 0$ and admit any $u \in V$ as admissible; however, as $\epsilon \to 0$, due to the "penalty term" $\epsilon^{-1}G(u)$, minimizers of E_ϵ will come closer and closer to M and, hopefully, will converge to a solution of the constrained minimization problem.

Even if $M = \emptyset$, minimizers u_ϵ of E_ϵ may still converge to a generalized solution of the Euler equations in some larger function space. This, however, requires controlling the penalty term $G(u_\epsilon)$. Sometimes this can be achieved via the following lemma.

7.2 Lemma. *Suppose that $E_\epsilon \colon V \to \mathbb{R}$, $0 < \epsilon < 1$, is a family of functionals with the following properties:*

(1°) For all $\epsilon \in \,]0,1[$ there exists $u_\epsilon \in V$ such that $E_\epsilon(u_\epsilon) = \inf_V E_\epsilon$.

(2°) For each $u \in V$ the map $\epsilon \mapsto E_\epsilon(u)$ is non-increasing.

Then for almost every $\epsilon_0 \in \,]0,1[$ the map

$$\epsilon \mapsto \nu_\epsilon = \inf_V E_\epsilon$$

is differentiable at ϵ_0 *with differential* ν'_{ϵ_0}*, and there holds*

$$\left| \frac{\partial E_\epsilon}{\partial \epsilon} \mid_{\epsilon=\epsilon_0} (u_{\epsilon_0}) \right| \leq |\nu'_{\epsilon_0}|.$$

The same conclusion holds if, instead of (2°) *, we assume that for each* $u \in V$ *the map* $\epsilon \mapsto E_\epsilon(u)$ *is non-decreasing.*

Proof. Since $\epsilon \mapsto E_\epsilon(u)$ is monotone, so is the map $\epsilon \mapsto \nu_\epsilon$. By Rademacher's theorem, the latter map therefore is differentiable at almost every $\epsilon_0 \in]\,0, 1\,[$. Finally, at any such ϵ_0, for $\epsilon > \epsilon_0$ by assumption (2°) we can estimate

$$0 \leq E_{\epsilon_0}(u_{\epsilon_0}) - E_\epsilon(u_{\epsilon_0}) \leq \nu_{\epsilon_0} - \nu_\epsilon.$$

Dividing by $\epsilon - \epsilon_0$, and passing to the limit $\epsilon \to \epsilon_0$, we obtain the claim. □

In the context of the penalty method described above, assumption (2°) of the lemma is always satisfied. Existence of a minimizer u_ϵ for E_ϵ, that is, assumption (1°) of the lemma, can be verified under very general conditions on E and G; compare Section 1. Thus, Lemma 7.2 applies. As an illustration we first state an abstract result in this regard. Later we will also give a concrete application.

Theorem 7.3. *Suppose* V *is reflexive, and let* $E, G\colon V \to \mathbb{R}$ *be non-negative and weakly lower semi-continuous. Also suppose that* E *is coercive. Then, for any* $\epsilon > 0$ *the functional* $E_\epsilon = E + \epsilon^{-1}G$ *attains its infimum* ν_ϵ *on* V*. Assume that* $\nu_\epsilon = o(|\ln \epsilon|)$ *as* $\epsilon \to 0$*. Then there exists a sequence* $\epsilon_n \to 0$ *such that* $\epsilon_n^{-1} G(u_{\epsilon_n}) \to 0$ *as* $n \to \infty$ *for any sequence* (u_{ϵ_n}) *of minimizers of* E_{ϵ_n}*.*

Proof. Existence of a minimizer u_ϵ of E_ϵ for $\epsilon > 0$ follows from Theorem 1.2. Moreover, for any fixed $\epsilon_0 > 0$ and $0 < \epsilon_1 < \epsilon_0$ by assumption on ν_ϵ we can estimate

$$o(|\ln \epsilon_1|) \geq \nu_{\epsilon_1} - \nu_{\epsilon_0} \geq \int_{\epsilon_1}^{\epsilon_0} |\nu'_\epsilon| d\epsilon \geq \operatorname*{ess\,inf}_{\epsilon_1 < \epsilon < \epsilon_0} \epsilon |\nu'_\epsilon| \cdot |\ln(\epsilon_1/\epsilon_0)|, \tag{7.1}$$

whence upon dividing by $|\ln \epsilon_1|$, and letting $\epsilon_1 \to 0$ in (7.1), we infer that

$$\liminf_{\epsilon \to 0} \epsilon |\nu'_\epsilon| = 0\ .$$

Thus, by Lemma 7.2, for a suitable sequence $\epsilon_n \to 0$ and any sequence (u_{ϵ_n}) of minimizers of E_{ϵ_n} there holds

$$\epsilon_n^{-1} G(u_{\epsilon_n}) = \epsilon_n \left| \frac{\partial E_{\epsilon_n}}{\partial \epsilon}(u_{\epsilon_n}) \right| \leq \epsilon_n |\nu'_{\epsilon_n}| \to 0 \, (n \to \infty),$$

as claimed. □

Similarly, if $M \neq \emptyset$, by the same arguments as above we can show that for a sequence $\epsilon_n \to 0$ there are minimizers (u_{ϵ_n}) of E_{ϵ_n} such that, as $n \to \infty$, (u_{ϵ_n}) converges weakly to a minimizer u_0 of E in M and the penalty term decays logarithmically as $\epsilon \to 0$; that is,

$$\epsilon_n^{-1} G(u_{\epsilon_n}) \leq C |\ln \epsilon_n|^{-1}$$

with a uniform constant C.

Harmonic Maps with Singularities

Let $\Omega \subset \mathbb{R}^2$ be smooth and bounded and assume (for simplicity) that Ω is simply connected. We can identify $\partial\Omega \cong S^1$ via a parametrization of the boundary curve.

Given a smooth map $u_0 \colon \partial\Omega \to S^1 \subset \mathbb{R}^2$, let

$$H^{1,2}_{u_0}(\Omega; \mathbb{R}^2) = \left\{ u \in H^{1,2}(\Omega; \mathbb{R}^2); u = u_0 \text{ on } \partial\Omega \right\} .$$

We seek to find a minimizer $u \in H^{1,2}_{u_0}(\Omega; \mathbb{R}^2)$ of Dirichlet's integral

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx$$

subject to the constraint

$$|u| = 1 ,$$

that is, a harmonic map $u \colon \Omega \to S^1$ with prescribed Dirichlet data $u = u_0$ on $\partial\Omega$. However, if the topological degree d of u_0, considered as a map $u_0 \colon S^1 \to S^1$, is non-zero, there is no extension of u_0 to a harmonic map $u \colon \Omega \to S^1$ of class $H^{1,2}$. In fact, by a result of Hélein [1] and its generalization to manifolds with boundary by Qing [1] such a map would be smooth. But, if $d \neq 0$, clearly there cannot be a C^1-map $u \colon \bar{\Omega} \to S^1$ such that $u = u_0$ on $\partial\Omega$. That is, any extension of u_0 to a (weakly) harmonic map must have singularities in $\bar{\Omega}$.

7.4 The Ginzburg Landau model. Bethuel-Brezis-Hélein proposed to study the resulting singular variational problem via the Ginzburg-Landau model. That is, for $\epsilon > 0$ they study the minimizers of the functional

$$E_\epsilon(u) = E(u) + \epsilon^{-2} G(u) , \qquad u \in H^{1,2}_{u_0}(\Omega; \mathbb{R}^2) ,$$

where

$$G(u) = \frac{1}{4} \int_\Omega (1 - |u|^2)^2 \, dx$$

penalizes the violation of the target constraint.

(Choosing ϵ^{-2} instead of ϵ^{-1} gives $\epsilon > 0$ a meaning as a "characteristic length" for the penalized problem; in our context, however, this is of no importance.) In a series of papers, summarized in their monograph Bethuel-Brezis-Hélein [1], they study the properties of minimizers of the Ginzburg-Landau energy and their convergence to a harmonic map with singularities. Moreover, they derive a renormalized energy functional whose minimizers give the possible locations of the singularities of the limiting harmonic map.

Three estimates, in particular, are needed for the proof of the convergence result. Let $u_\epsilon \in H^{1,2}_{u_0}(\Omega;\mathbb{R}^2)$ be minimizers of E_ϵ for $\epsilon > 0$. We may suppose $d > 0$. Constants may depend on Ω and u_0.

It is rather straightforward to establish the following upper bound

$$E_\epsilon(u_\epsilon) \le \pi d|\ln\epsilon| + C\ . \tag{7.2}$$

Establishing the corresponding lower bound

$$E_\epsilon(u_\epsilon) \ge \pi d|\ln\epsilon| - C \tag{7.3}$$

is considerably more difficult, while the final estimate

$$\limsup_{\epsilon\to 0} \epsilon^{-2} G(u_\epsilon) \le C \tag{7.4}$$

for the convergence proof seems to be the most delicate. Using "Pohožaev's identity", Bethuel-Brezis-Hélein proved this estimate for star-shaped domains; compare Lemma III.1.4. For general domains, the result is due to Struwe [22], [23].

For *a sequence* $\epsilon_n \to 0$, estimate (7.4) is an immediate consequence of the "easy" upper bound (7.2) and Lemma 7.1; see Struwe [22].

7.5 Lemma. *Suppose the estimate (7.2) holds true with a uniform constant C. Then we have*

$$\liminf_{\epsilon\to 0} \epsilon^{-2} G(u_\epsilon) \le \pi d.$$

Proof. Let $\nu_\epsilon = \inf_{H^{1,2}_{u_0}(\Omega;\mathbb{R}^2)} E_\epsilon \ge 0$. From (7.2) and Lemma 7.2, similar to (7.1) we deduce that

$$\begin{aligned}
\pi d|\ln\epsilon| + C &\ge \int_\epsilon^1 (\epsilon|\nu'_\epsilon|)\frac{d\epsilon}{\epsilon} \\
&\ge \int_\epsilon^1 \left|\epsilon\frac{\partial E_\epsilon}{\partial\epsilon}(u_\epsilon)\right| \frac{d\epsilon}{\epsilon} \\
&= 2\int_\epsilon^1 \big(\epsilon^{-2}G(u_\epsilon)\big)\frac{d\epsilon}{\epsilon}\ ,
\end{aligned}$$

and the assertion follows. □

In combination with the lower bound (7.3), the same ideas actually yield (7.4) for *any* sequence of minimizers. This observation is due to del Pino-Felmer [1].

7.6 Lemma. *Suppose the estimates (7.2) and (7.3) hold true with a uniform constant C. Then we have*

$$\limsup_{\epsilon \to 0} \epsilon^{-2} G(u_\epsilon) \leq C.$$

Proof. Instead of differentiating, we now take finite differences. Taking a minimizer u_ϵ of E_ϵ as comparison function for $E_{2\epsilon}$ and using (7.2), (7.3), we obtain

$$\begin{aligned} \frac{3}{4}\epsilon^{-2} G(u_\epsilon) &= E_\epsilon(u_\epsilon) - E_{2\epsilon}(u_\epsilon) \\ &\leq \pi d |\ln \epsilon| + C - (\pi d |\ln(2\epsilon)| - C) \\ &\leq \pi d \ln 2 + C \ , \end{aligned}$$

as claimed. □

Thus, the proof of estimate (7.4) is complete. The arguments of Bethuel-Brezis-Hélein then finally yield the following convergence result.

7.7 Theorem. *For any sequence $\epsilon_n \to 0$ and any sequence of minimizers u_n of E_{ϵ_n} there exist exactly d points $x_1, \ldots, x_d \in \Omega$ such that, as $n \to \infty$, a subsequence $u_n \rightharpoonup \bar{u}$ weakly in $H^{1,2}_{loc}(\Omega \setminus \{x_1, \ldots, x_d\}; \mathbb{R}^2)$ and in $H^{1,p}(\Omega, \mathbb{R}^2)$ for any $p < 2$.*

The limit map $\bar{u}\colon \Omega \setminus \{x_1, \ldots, x_d\} \to S^1$ is harmonic and is given by

$$\bar{u}(x) = e^{i\phi(x)} \prod_{j=1}^{d} \frac{x - x_j}{|x - x_j|}$$

with some harmonic function ϕ. (We identify $\mathbb{R}^2 \cong \mathbb{C}$.)

The singularities $x_1, \ldots, x_d$ are minima of a renormalized energy functional given explicitly in terms of u_0 and the Green's function on Ω.

7.8 Notes. The idea of using variations of parameters in order to obtain a-priori bounds on critical points (in fact, saddle points) of variational integrals depending monotonically on a real parameter was introduced by Struwe [16], in the context of surfaces of prescribed mean curvature $H \in \mathbb{R}$. Further applications are given in Struwe [17], [21], and Ambrosetti-Struwe[2]. Related ideas have been proposed by Schechter-Tintarev [1]. See the Preface to the Third Edition for more recent developments and references.

In Section II.9 we return to the topic of parameter dependence in the context of saddle points and give applications of Hamiltonian systems.

Chapter II

Minimax Methods

In the preceding chapter we have seen that (weak sequential) lower semicontinuity and (weak sequential) compactness of the sub-level sets of a functional E on a Banach space V suffice to guarantee the existence of a minimizer of E.

To prove the existence of saddle points we will now strengthen the regularity hypothesis on E and in general require E to be of class $C^1(V)$, that is continuously Fréchet differentiable. In this case, the notion of critical point is defined and it makes sense to classify such points as relative minima or saddle points as we did in the introduction to Chapter I.

Moreover, we will impose a certain compactness assumption on E, to be stated in Section 2. First, however, we recall a classical result in finite dimensions.

1. The Finite Dimensional Case

In the finite dimensional case, the existence of saddle points can be obtained for instance as follows (see, for instance, Courant [1; p. 223 ff.]):

1.1 Theorem. *Suppose $E \in C^1(\mathbb{R}^n)$ is coercive and suppose that E possesses two distinct strict relative minima x_1 and x_2. Then E possesses a third critical point x_3 which is not a relative minimizer of E and hence distinct from x_1, x_2, characterized by the minimax principle*

$$E(x_3) = \inf_{p \in P} \max_{x \in p} E(x) =: \beta \ ,$$

where

$$P = \{p \subset \mathbb{R}^n \ ; \ x_1, x_2 \in p, \ p \text{ is compact and connected}\}$$

is the class of "paths" connecting x_1 and x_2.

Proof. Let (p_m) be a minimizing sequence in P, satisfying

$$\max_{x \in p_m} E(x) \to \beta \qquad (m \to \infty) \ .$$

Since E is coercive, the sets p_m are uniformly bounded. Therefore

$$p = \bigcap_{m\in\mathbb{N}} \overline{\bigcup_{l\geq m} p_l} \ ,$$

the set of accumulation points of (p_m), is the intersection of a decreasing sequence of compact and connected sets, hence is compact and connected. Moreover, by construction $x_1, x_2 \in p_m$ for every m. Hence $x_{1,2} \in p$, and $p \in P$. Thus

$$\max_{x\in p} E(x) \geq \inf_{p'\in P} \max_{x\in p'} E(x) = \beta \ .$$

By continuity, on the other hand,

$$\max_{x\in p} E(x) \leq \limsup_{m\to\infty} (\max_{x\in p_m} E(x)) = \beta \ ,$$

and $\max_{x\in p} E(x) = \beta$.

Note that, in particular, since $x_{1,2}$ are strict relative minima joined by p, it now also follows that $\beta > \max\{E(x_1), E(x_2)\}$.

To see that there is a critical point $x_3 \in p$ such that $E(x_3) = \beta$, we argue indirectly: Note that by continuity of E and compactness of p the set $K = \{x \in p \ ; \ E(x) = \beta\}$ is compact. Suppose $DE(x) \neq 0$ for every $x \in K$. Then there is a uniform number $\delta > 0$ such that $|DE(x)| \geq 2\delta$ for all $x \in K$.

By continuity, there exists a neighborhood

$$U_\varepsilon = \{x \in \mathbb{R}^n \ ; \ \exists y \in K : |x - y| < \varepsilon\}$$

of K such that $|DE(x)| \geq \delta$ in U_ε. Note that this implies $x_1, x_2 \notin U_\varepsilon$. Let η be a continuous cut-off function with support in U_ε such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in a neighborhood of K. Let $\nabla E(x)$ denote the gradient of E at x, characterized by the condition

$$\nabla E(x) \cdot v = DE(x)v \ \text{ for all } v \in \mathbb{R}^n \ .$$

Define a continuous map $\Phi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by letting

$$\Phi(x,t) = x - t\eta(x) \ \nabla E(x) \ .$$

Note that Φ is continuously differentiable in t and

$$\frac{d}{dt} E\left(\Phi(x,t)\right)\Big|_{t=0} = - < \eta(x)\nabla E(x), DE(x) > = -\eta(x)|\nabla E(x)|^2.$$

Moreover, $|\nabla E(x)|^2 \geq \delta^2 > 0$ on $\mathrm{supp}(\eta) \subset U_\varepsilon$. By continuity then, there exists $T > 0$ such that

$$\frac{d}{dt} E\left(\Phi(x,t)\right) \leq -\frac{\eta(x)}{2}|\nabla E(x)|^2$$

for all $t \in [0,T]$, uniformly in x. Thus if we choose

$$p_T = \{\Phi(x,T) \, ; \ x \in p\} \ ,$$

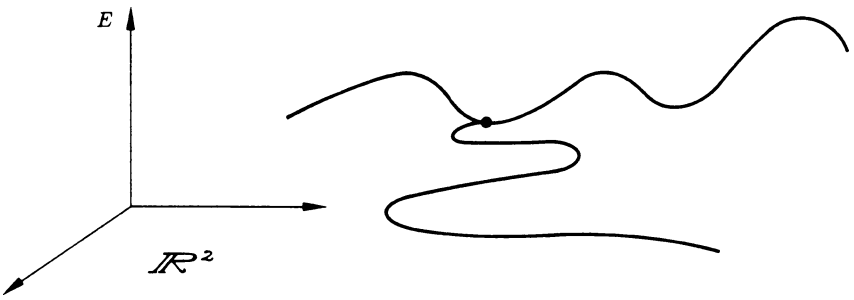

Fig. 1.1. A mountain pass in the E-landscape

for any point $\Phi(x,T) \in p_T$ we compute that

$$\begin{aligned} E\left(\Phi(x,T)\right) &= E(x) + \int_0^T \frac{d}{dt} E\left(\Phi(x,t)\right)\, dt \\ &\leq E(x) - \frac{T}{2}\eta(x)\, |\nabla E(x)|^2 \,, \end{aligned}$$

and the latter is either $\leq E(x) < \beta$, if $x \notin K$, or $\leq \beta - \frac{T}{2}\delta^2 < \beta$, if $x \in K$. Hence

$$\max_{x \in p_T} E(x) < \beta \,.$$

But by continuity of Φ it follows that p_T is compact and connected, while by choice of U_ε and η also $x_i = \Phi(x_i, T) \in p_T$, $i = 1, 2$. Hence $p_T \in P$, contradicting the definition of β.

Finally, if all critical points u of E in p with $E(u) = \beta$ were relative minima, the set $\tilde{K}$ of such points would be open in p and (by continuity of E and DE) also closed. Moreover, by the preceding argument $\tilde{K} \neq \emptyset$. But p is connected. Thus $p = \tilde{K}$, contradicting the fact that $E(x_1), E(x_2) < \beta$. This concludes the proof. □

1.2 Interpretation. It is useful to think of $E(x)$ as measuring the elevation at a point x in a landscape. Our two minima x_1, x_2 then correspond to two villages at the deepest points of two valleys, separated from each other by a mountain ridge. If now we walk along a path p from x_1 to x_2 with the property that the maximal elevation $E(x)$ at points x on p is minimal among all such paths we will cross the ridge at a mountain pass x_3 which is a saddle point of E. Because of this geometric interpretation Theorem 1.1 is sometimes called the finite dimensional "mountain pass theorem".

2. The Palais-Smale Condition

From the experience in the preceding section we expect a functional to possess critical points of saddle type whenever the set of points with energy less than a certain value is disconnected or has a non-trivial topology. However, even in the finite dimensional setting of Theorem 1.1 and with suitable assumptions about the topology of the sub-level sets of E, saddle points in general need not exist unless a certain compactness property holds. This is illustrated by the following simple example.

Example. Let $E \in C^1(\mathbb{R}^2)$ be given by $E(x,y) = exp(-y) - x^2$. Then $E_0 := \{(x,y)\,;\ E(x,y) < 0\}$ is disconnected, while there is no "mountain pass" of minimal height 0; see Figure 2.1. □

Note, however, that in the example above there is a sequence of paths p_m, given by $p_m(t) = (t,m),\ -1 \le t \le 1$, connecting the two components of E_0, such that E achieves its maximum on p_m at points $z_m = (0,m)$, satisfying $E(z_m) \to 0,\ DE(z_m) \to 0$ as $m \to \infty$. Moreover, the points (z_m) fail to have a finite accumulation point. It seems that this lack of compactness is responsible for the absence of saddle-type critical points.

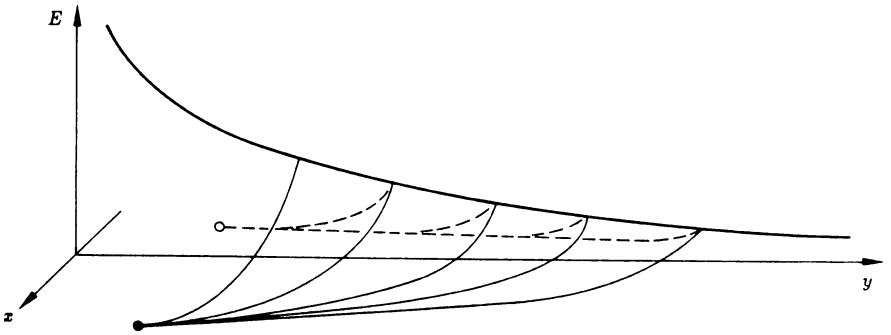

Fig. 2.1. Searching in vain for an optimal mountain pass

As we have seen, one way of inducing the necessary compactness in the finite dimensional case is by requiring E to be coercive, which generalizes to the condition of bounded compactness in the infinite dimensional case.

However, as remarked earlier, in infinite dimensions the requirements of bounded compactness and our regularity assumption $E \in C^1(V)$ are incompatible. Moreover, we would like to apply minimax methods to functionals which in general are neither bounded from above nor below. Thus, any of the

former conditions on E is too restrictive. Instead, we will require the so-called *Palais-Smale* condition to be satisfied by E. Originally, in the work of Palais and Smale this assumption is stated as follows:

(C) If S is a subset of V on which $|E|$ is bounded but on which $\|DE\|$ is not bounded away from zero, then there is a critical point in the closure of S.

(See Palais [1], [2], Smale [2], Palais-Smale [1].)

We will replace condition (C) by a slightly stronger condition which is easier to work with. It is convenient to introduce the following concept.

Definition. *A sequence (u_m) in V is a Palais-Smale sequence for E if $|E(u_m)| \leq c$, uniformly in m, while $\|DE(u_m)\| \to 0$ as $m \to \infty$.*

In terms of this definition our compactness condition may be phrased as follows.

(P.-S.) Any Palais-Smale sequence has a (strongly) convergent subsequence.

Condition (P.-S.) implies condition (C): From any set S as in condition (C), we may extract a Palais-Smale sequence, and, if the latter has a convergent subsequence, the limit point of this sequence will be a critical point in the closure of S. The converse, however, is not true: The functional $E \equiv 0$ satisfies (C) but in general will not satisfy (P.-S.).

Note that (P.-S.) implies that any set of critical points of uniformly bounded energy is relatively compact, see Lemma 2.3.(1°). In fact, if we were to strengthen condition (C) by this requirement, this new condition would be equivalent to (P.-S.).

In finite dimensions, a large class of functionals satisfying (P.-S.) can be characterized as follows:

2.1 Proposition. *Suppose $E \in C^1(\mathbb{R}^n)$ and assume the function $\|DE\| + |E| : \mathbb{R}^n \to \mathbb{R}$ is coercive. Then (P.-S.) holds for E.*

Proof. If $\|DE\| + |E|$ is coercive, clearly a Palais-Smale sequence will be bounded, hence will contain a convergent subsequence by the Bolzano-Weierstrass theorem. □

Example. Suppose $E : \mathbb{R}^n \to \mathbb{R}$ is a quadratic polynomial

$$E(x) = \sum_{i,j=1}^{n} a_{ij}\, x_i\, x_j \;+\; \sum_{i=1}^{n} b_i\, x_i \;+\; c$$

in $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and that $D^2E(x) = (a_{ij})_{1 \le i,j \le n}$ is non-degenerate in the sense that the matrix $(a_{ij})_{1 \le i,j \le n}$ induces an invertible linear map $\mathbb{R}^n \to \mathbb{R}^n$. Then E satisfies (P.-S.).

We may ask whether a similar non-degeneracy condition in general will guarantee that a polynomial map satisfies (P.-S.). Suppose E is a polynomial of degree m in $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$E(x) = \sum_{|\alpha| \le m} a_\alpha x^\alpha \ ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Suppose $D^2E(x) \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is non-degenerate for any $x \in \mathbb{R}^n$. Does E satisfy (P.-S.)? (The answer seems to be unknown. In fact, this question seems related to a particular case of the Jacobi conjecture, a puzzling problem in algebraic geometry; see, for instance, Bass-Connell-Wright [1].)

In general, we can say the following:

2.2 Proposition. *Suppose that E has the following properties.*
(1°) Any Palais-Smale sequence for E is bounded in V.
(2°) For any $u \in V$ we can decompose

$$DE(u) = L + K(u) \ ,$$

where $L \colon V \to V^*$ is a fixed boundedly invertible linear map and the operator K maps bounded sets in V to relatively compact sets in V^*. Then E satisfies (P.-S.).

Proof. Any (P.-S.)-sequence (u_m) is bounded by assumption. Moreover

$$DE(u_m) = Lu_m + K(u_m) \to 0$$

implies that

$$u_m = o(1) - L^{-1}K(u_m) \ ,$$

where $o(1) \to 0$ in V as $m \to \infty$. By boundedness of (u_m) and compactness of K the sequences $(L^{-1}K(u_m))$ – and hence (u_m) – are relatively compact. □

The Palais-Smale condition permits one to distinguish a certain family of neighborhoods of critical points of a functional E and thus offers a useful characterization of regular values of E.

For $\beta \in \mathbb{R}$, $\delta > 0$, $\rho > 0$ let

$$\begin{aligned} E_\beta &= \{u \in V \ ; \ E(u) < \beta\}, \\ K_\beta &= \{u \in V \ ; \ E(u) = \beta, \ DE(u) = 0\} \ , \\ N_{\beta,\delta} &= \{u \in V \ ; \ |E(u) - \beta| < \delta, \ \|DE(u)\| < \delta\} \ , \\ U_{\beta,\rho} &= \bigcup_{u \in K_\beta} \{v \in V \ ; \ \|v - u\| < \rho\} \ . \end{aligned}$$

That is, K_β is the set of critical points of E having "energy" β, $\{U_{\beta,\rho}\}_{\rho>0}$ is the family of norm-neighborhoods of K_β.

2.3 Lemma. *Suppose E satisfies (P.-S.). Then for any $\beta \in \mathbb{R}$ the following holds:*
(1°) K_β is compact.
(2°) The family $\{U_{\beta,\delta}\}_{\rho>0}$ is a fundamental system of neighborhoods of K_β.
(3°) The family $\{N_{\beta,\delta}\}_{\delta>0}$ is a fundamental system of neighborhoods of K_β.

Proof. (1°) Any sequence (u_m) in K_β by (P.-S.) has a convergent subsequence. By continuity of E and DE any accumulation point of such a sequence also lies in K_β, and K_β is compact.
(2°) Any $U_{\beta,\rho}$, $\rho > 0$, is a neighborhood of K_β. Conversely, let N be any open neighborhood of K_β. Suppose by contradiction that for $\rho_m \to 0$ there is a sequence of points $u_m \in U_{\beta,\rho_m} \setminus N$. Let $v_m \in K_\beta$ be such that $\|u_m - v_m\| \le \rho_m$. Since K_β is compact by (1°), we may assume that $v_m \to v \in K_\beta$. But then also $u_m \to v$, and $u_m \in N$ for large m, contrary to assumption.
(3°) Similarly, each $N_{\beta,\delta}$, for $\delta > 0$, is a neighborhood of K_β. Conversely, suppose that for some neighborhood N of K_β and $\delta_m \to 0$ there is a sequence $u_m \in N_{\beta,\delta_m} \setminus N$. By (P.-S.) the sequence (u_m) accumulates at a critical point $u \in K_\beta \subset N$. The contradiction proves the lemma. □

2.4 Remarks. (1°) In particular, if $K_\beta = \emptyset$ for some $\beta \in \mathbb{R}$ there exists $\delta > 0$ such that $N_{\beta,\delta} = \emptyset$; that is, the differential $DE(u)$ is uniformly bounded in norm away from 0 for all $u \in V$ with $E(u)$ close to β.
(2°) The conclusion of Lemma 2.3 remains valid at the level β under the weaker assumption that (P.-S.)-sequences (u_m) for E such that $E(u_m) \to \beta$ are relatively compact. This observation will be useful when dealing with limiting cases for (P.-S.) in Chapter III.

2.5 Cerami's variant of (P.-S.). Cerami [1], [2; Teorema (*), p. 166] has proposed the following variant of (P.-S.):

(2.1) Any sequence (u_m) such that $|E(u_m)| \le c$ uniformly and $\|DE(u_m)\|\big(1 + \|u_m\|\big) \to 0$ $(m \to \infty)$ has a (strongly) convergent subsequence.

Condition (2.1) is slightly weaker than (P.-S.) while the most important implications of (P.-S.) are retained; see Cerami [1],[2] or Bartolo-Benci-Fortunato [1]. However, for most purposes it suffices to use the standard (P.-S.) condition. Therefore and in order to achieve a coherent and simple exposition consistent with the bulk of the literature in the field, in the following our presentation will be based on (P.-S.) rather than (2.1).

Variants of the Palais-Smale condition for non-differentiable functionals will be discussed in a later section. (See Section 10.)

3. A General Deformation Lemma

Besides the compactness condition, the second main ingredient in the proof of Theorem 1.1 is the (local) gradient-line deformation Φ. Following Palais [4], we will now construct a similar deformation for a general C^1-functional in a Banach space. The construction may be carried out in the more general setting of C^1-functionals on complete, regular $C^{1,1}$-Banach manifolds with Finsler structures; see Palais [4]. However, for most of our purposes it suffices to consider a Banach space as the ambient space on which a functional is defined.

Pseudo-gradient Flows on Banach Spaces

In the following we will initially assume that E is a C^1-functional on a Banach space V. Moreover, we denote

$$\tilde{V} = \{u \in V \ ; \ DE(u) \neq 0\}$$

the set of regular points of E. As a substitute for the notion of gradient (which requires an inner product to be defined) we introduce the following concept.

3.1 Definition. *A pseudo-gradient vector field for E is a locally Lipschitz continuous vector field $v\colon \tilde{V} \to V$ such that the conditions*
(1°) $\|v(u)\| < 2\min\{\|DE(u)\|, 1\}$,
(2°) $\langle v(u), DE(u)\rangle \ > \min\{\|DE(u)\|, 1\}\,\|DE(u)\|$
hold for all $u \in \tilde{V}$.

Note that we require v to be locally Lipschitz. Hence, even in a Hilbert space the following result is somewhat remarkable.

3.2 Lemma. *Any functional $E \in C^1(V)$ admits a pseudo-gradient vector field $v\colon \tilde{V} \to V$.*

Proof. For $u \in \tilde{V}$ choose $w = w(u)$ such that

$$\|w\| < 2\min\{\|DE(u)\|, 1\} \ , \tag{3.1}$$

$$\langle w, DE(u)\rangle \ > \ \min\{\|DE(u)\|, 1\}\|DE(u)\| \ . \tag{3.2}$$

By continuity, for any $u \in \tilde{V}$ there is a neighborhood $W(u)$ such that (3.1) and (3.2) hold (with $w = w(u)$) for all $u' \in W(u)$. Since $\tilde{V} \subset V$ is metrizable and hence paracompact, there exists a locally finite refinement $\{W_\iota\}_{\iota\in I}$ of the cover $\{W(u)\}_{u\in\tilde{V}}$ of $\tilde{V}$, consisting of neighborhoods $W_\iota \subset W(u_\iota)$; see for instance Kelley [1; Corollary 5.35, p. 160].

Choose a Lipschitz continuous partition of unity $\{\varphi_\iota\}_{\iota\in I}$ subordinate to $\{W_\iota\}_{\iota\in I}$; that is, choose Lipschitz continuous functions $0 \leq \varphi_\iota \leq 1$ with support in W_ι and such that $\sum_{\iota\in I} \varphi_\iota \equiv 1$ on $\tilde{V}$. For instance, following Palais [4; p. 205 f.], we may let

$$\rho_\iota(u) = dist(u, V \setminus W_\iota) = \inf\{\|u - v\| \ ; \ v \notin W_\iota\}$$

and define

$$\varphi_\iota(u) = \frac{\rho_\iota(u)}{\sum_{\iota' \in I} \rho_{\iota'}(u)} \ .$$

Clearly, $0 \le \varphi_\iota \le 1$, $\varphi_\iota = 0$ outside W_ι, and $\sum_{\iota \in I} \varphi_\iota \equiv 1$. Moreover, since $\{W_\iota\}_{\iota \in I}$ is locally finite, for any $u \in \tilde{V}$ there exists a neighborhood W of u such that $W \cap W_{\iota'} \neq \emptyset$ for at most finitely many indices $\iota' \in I$, and Lipschitz continuity of φ_ι on W is immediate from Lipschitz continuity of the family $\{\rho_\iota\}_{\iota \in I}$.

Finally, we may let

$$v(u) = \sum_{\iota \in I} \varphi_\iota(u) \ w(u_\iota) \ .$$

The relations (3.1) and (3.2) being satisfied by $w(u_\iota)$ on the support of φ_ι, their convex linear combination v is a pseudo-gradient vector field for E, as required. □

3.3 Remark. If E admits some compact group action G as symmetries, v may be constructed to be G-equivariant. In particular, if E is even, that is, $E(u) = E(-u)$, with symmetry group $\{id, -id\} \cong \mathbb{Z}_2$, we may choose

$$\tilde{v}(u) = \frac{1}{2}\left(v(u) - v(-u)\right) \ ,$$

where v is any pseudo-gradient vector field for E, to obtain a $\mathbb{Z}_2$-equivariant pseudo-gradient vector field v for E, satisfying $v(-u) = -v(u)$.

More generally, suppose G is a compact Lie group acting on V; that is, suppose there is a group homomorphism of G onto a subgroup – indiscriminately denoted by G – of the group of linear isometries of V such that the evaluation map

$$G \times V \to V \ ; \ (g, u) \mapsto gu$$

is continuous. Also suppose that G leaves E invariant

$$E(gu) = E(u), \qquad \forall (g, u) \in G \times V \ .$$

Then it suffices to let

$$\tilde{v}(u) = \int_G g^{-1} v(gu) \, dg$$

be the average of any pseudo-gradient vector field v for E with respect to an invariant Haar's measure dg on G in order to obtain a G-equivariant pseudo-gradient vector field $\tilde{v}$, satisfying

$$\tilde{v}(gu) = g\tilde{v}(u)$$

for all g and u.

We are now ready to state the main theorem in this section.

3.4 Theorem (Deformation Lemma). *Suppose $E \in C^1(V)$ satisfies (P.-S.). Let $\beta \in \mathbb{R}$, $\overline{\varepsilon} > 0$ be given and let N be any neighborhood of K_β. Then there exist a number $\varepsilon \in]0, \overline{\varepsilon}[$ and a continuous 1-parameter family of homeomorphisms $\Phi(\cdot, t)$ of V, $0 \le t < \infty$, with the properties*
(1°) $\Phi(u,t) = u$, if $t = 0$, or $DE(u) = 0$, or $|E(u) - \beta| \ge \overline{\varepsilon}$;
(2°) $E(\Phi(u,t))$ is non-increasing in t for any $u \in V$;
(3°) $\Phi(E_{\beta+\varepsilon} \setminus N, 1) \subset E_{\beta-\varepsilon}$, and $\Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N$.
Moreover, $\Phi: V \times [0,\infty[\to V$ has the semi-group property; that is, $\Phi(\cdot, t) \circ \Phi(\cdot, s) = \Phi(\cdot, s+t)$ for all $s, t \ge 0$.

Proof. Lemma 2.3 permits one to choose numbers $\delta, \rho > 0$ such that

$$N \supset U_{\beta,2\rho} \supset U_{\beta,\rho} \supset N_{\beta,\delta} \ .$$

We may suppose $\delta, \rho \le 1$. Let η be a locally Lipschitz continuous function on V such that $0 \le \eta \le 1$, $\eta \equiv 1$ outside $N_{\beta,\delta}$, $\eta \equiv 0$ in $N_{\beta,\delta/2}$. Also let φ be a Lipschitz continuous function on $\mathbb{R}$ such that $0 \le \varphi \le 1$, $\varphi(s) \equiv 0$, if $|\beta - s| \ge \min\{\overline{\varepsilon}, \delta/4\}$, $\varphi(s) \equiv 1$, if $|\beta - s| \le \min\{\overline{\varepsilon}/2, \delta/8\}$. Finally, let $v: \tilde{V} \to V$ be a pseudo-gradient vector field for E. Define

$$e(u) = \begin{cases} -\eta(u)\ \varphi(E(u))\, v(u), & \text{if } u \in \tilde{V} \\ 0, & \text{else} \end{cases} \ .$$

By choice of φ and η, the vector field e vanishes identically (and therefore is Lipschitz continuous) near critical points u of E. Hence e is locally Lipschitz continuous throughout V. Moreover, since $\|v\| < 2$ uniformly, also $\|e\| \le 2$ is uniformly bounded. Hence there exists a global solution $\Phi: V \times \mathbb{R} \to V$ of the initial value problem

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(u,t) &= e(\Phi(u,t)) \\ \Phi(u,0) &= u \ . \end{aligned}$$

Φ is continuous in u, differentiable in t and has the semi-group property $\Phi(\cdot, s) \circ \Phi(\cdot, t) = \Phi(\cdot, s+t)$, for any s, $t \in \mathbb{R}$. In particular, for any $t \in \mathbb{R}$ the map $\Phi(\cdot, t)$ is a homeomorphism of V.

Properties (1°) and (2°) are trivially satisfied by construction and the properties of v. Moreover, for $\varepsilon \le \min\{\overline{\varepsilon}/2, \delta/8\}$ and $u \in E_{\beta+\varepsilon}$, if $E(\Phi(u,1)) \ge \beta - \varepsilon$ it follows from (2°) that $|E(\Phi(u,t)) - \beta| \le \varepsilon$ and hence that $\varphi\big(E\big(\Phi(u,t)\big)\big) = 1$ for all $t \in [0,1]$.

Differentiating, by the chain rule we thus obtain

$$\begin{aligned} E\big(\Phi(u,1)\big) &= E(u) + \int_0^1 \frac{d}{dt} E(\Phi(u,t))\ dt \\ &< \beta + \varepsilon - \int_0^1 \eta(\Phi(u,t)) \langle v(\Phi(u,t)), DE(\Phi(u,t)) \rangle\, dt \\ &\le \beta + \varepsilon - \int_{\{t; \Phi(u,t) \notin N_{\beta,\delta}\}} \langle v(\Phi(u,t)), DE(\Phi(u,t)) \rangle\, dt \\ &\le \beta + \varepsilon - \mathcal{L}^1\big(\{t; \Phi(u,t) \notin N_{\beta,\delta}\}\big) \cdot \delta^2 \ . \end{aligned} \tag{3.3}$$

But if either $u \notin N$ or $\Phi(u,1) \notin N$, by uniform boundedness $\|e\| \le 2$ and since $V \setminus N$ and $N_{\beta,\delta}$ are separated by the "annulus" $U_{\beta,2\rho} \setminus U_{\beta,\rho}$ of width ρ, certainly

$$\mathcal{L}^1\big(\{t\ ;\ \Phi(u,t) \notin N_{\beta,\delta}\}\big) \ge \frac{\rho}{2}\ . \tag{3.4}$$

Hence, if we choose $\varepsilon \le \frac{\delta^2\rho}{4}$, estimate (3.3) gives

$$E\left(\Phi(u,1)\right) < \beta + \varepsilon - \frac{\rho\delta^2}{2} \le \beta - \varepsilon\ ,$$

and (3°) follows. □

3.5 Remarks. (1°) Since the deformation $\Phi\colon V \times [0,\infty[\to V$ is obtained by integrating a suitably truncated pseudo-gradient vector field, Φ will be called a (local) pseudo-gradient flow.
(2°) If $K_\beta = \emptyset$, we may choose $N = \emptyset$ and hence obtain a uniform reduction of energy near β in this case.
(3°) The Palais-Smale condition was only used to obtain estimate (3.4). For this it is enough to assume that (P.-S.) holds at the level β, see Remark 2.4(2°). In particular, if $N = K_\beta = \emptyset$ the conclusion of Theorem 3.4 remains valid if condition (P.-S.)is replaced by the assumption that $N_{\beta,\delta} = \emptyset$ for some $\delta > 0$.
(4°) If E is invariant under a compact group action G, as in Remark 3.3, we can achieve that Φ is G-equivariant in the sense that there holds

$$\Phi(gu,t) = g\,\Phi(u,t) \qquad \text{for all } u \in V,\ g \in G,\ t \ge 0\ .$$

3.6 Comparison with gradient flows. It may be of interest to consider the special case of a C^2-functional E on a real Hilbert space H with scalar product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$. In this case, a gradient vector field $\nabla E\colon H \to H$ is defined as in the finite dimensional case by letting $\nabla E(u)$ at any $u \in H$ be the unique vector in H such that $\big(\nabla E(u), v\big) = DE(u)v$ for all $v \in H$, equivalently characterized by

$$\|\nabla E(u)\| = \|DE(u)\|,\ \langle \nabla E(u), DE(u)\rangle = \|DE(u)\|^2\ . \tag{3.5}$$

Moreover, since

$$\|\nabla E(u) - \nabla E(v)\| = \|DE(u) - DE(v)\|\ ,$$

if $E \in C^2,\ \nabla E$ is of class C^1 and defines a local gradient flow Φ by letting

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(u,t) &= -\nabla E\left(\Phi(u,t)\right) \\ \Phi(u,0) &= u\ . \end{aligned}$$

To interpret Φ we identify E with its graph $\mathcal{G}(E) = \{(u, E(u)) \in H \times \mathbb{R}\}$. Then in the picture outlined in Interpretation 1.2 the flow-lines of Φ become paths

of steepest descent, and the rest points of Φ are precisely the critical points of E where $\mathcal{G}(E)$ has a horizontal tangent plane.

In a Banach space V as ambient space, note that in general by (3.5) a gradient vector need not be uniquely determined unless V is uniformly locally convex. Moreover, also in this case, the duality map $j: V^* \to V$, which maps $v \in V^*$ to $w = j(v) \in V$ satisfying $\langle w, v\rangle = \|v\|^2 = \|w\|^2$, in general is only uniformly continuous on bounded sets but may fail to be Lipschitz.

Fortunately, it is not at all necessary to deform along lines of "steepest" descent to obtain existence results for saddle points, "steep enough" suffices. The notions of pseudo-gradient vector field and pseudo-gradient flow allow for the necessary flexibility.

Pseudo-Gradient Flows on Manifolds

The above constructions of pseudo-gradient vector fields and pseudo-gradient flows can easily be generalized to the setting of a C^1-functional on a complete $C^{1,1}$-Finsler manifold. We basically follow Palais [4].

3.7 Finsler manifolds. Let F be a Banach space bundle over a space M and let $\|\cdot\|$ be a continuous real valued function on F such that the restriction $\|\cdot\|_u$ of $\|\cdot\|$ to each fiber F_u is an admissible norm for F_u. If we trivialize F in a neighborhood of a point $u_0 \in M$, using F_{u_0} as the standard fiber, then for each u near u_0 the norm $\|\cdot\|_u$ becomes a norm on F_{u_0}. We say $\|\cdot\|$ is a Finsler structure for the bundle F if for any $\varepsilon > 0$, each $u_0 \in M$, and each such trivialization in an atlas defining the bundle structure of E

$$\sup_{v \in F_{u_0}\setminus\{0\}} \frac{\|v\|_u}{\|v\|_{u_o}}\ , \quad \sup_{v\in F\setminus\{0\}} \frac{\|v\|_{u_0}}{\|v\|_u} < 1+\varepsilon\ ,$$

if u is sufficiently near u_0.

Recall that a topological space M is regular if for each point $x \in M$ and any neighborhood U of u there is a closed neighborhood $\tilde{U}$ of u such that $\tilde{U} \subset U$. Now, a Finsler manifold of class C^r, $r \geq 1$, is a regular C^r-Banach manifold M, modeled on a Banach space V, together with a Finsler structure $\|\cdot\|$ on the tangent bundle TM. Then also the co-tangent space T^*M carries a natural Finsler structure, indiscriminately denoted by $\|\cdot\|$, characterized by letting

$$\|v^*\|_u = \sup\big\{|\langle v, v^*\rangle|\ ;\ v \in T_uM,\ \|v\|_u \leq 1\big\}$$

for any $v^* \in T_u^*M$. Finally, $\|\cdot\|$ induces a metric

$$d(u,v) = \inf_p \int_0^1 \left\| \frac{d}{dt} p(t)\right\|_{p(t)} dt\ , \tag{3.5}$$

where the infimum is taken over all C^1-paths $p\colon [0,1] \to M$ joining $p(0) = u$, $p(1) = v$; see Palais [4; p. 208 ff.]. We say that M is complete if M is complete with respect to this metric d.

As an example of a complete C^r-Finsler manifold we may consider any (norm-) complete C^r-submanifold M of a Banach space V, with T_uM carrying the norm induced by the inclusion $T_uM \subset T_uV \cong V$.

3.8 Definition. *Let M be a $C^{1,1}$ Finsler manifold modeled on a Banach space V, $E \in C^1(M)$, $\tilde{M} = \{u \in M \,;\, DE(u) \neq 0\}$ the set of regular points of E. A pseudo-gradient vector field for E is a Lipschitz continuous vector field (section) $v\colon \tilde{M} \mapsto TM$ in the tangent bundle TM with the property that $v(u) \in T_uM$ and*
(1°) $\|v(u)\|_u < 2\min\{\|DE(u)\|_u, 1\}$,
(2°) $\langle v(u), DE(u)\rangle \;>\; \min\{\|DE(u)\|_u, 1\}\|DE(u)\|_u$,
for all $u \in \tilde{M}$, where $\|\cdot\|_u$ denotes the norm in the tangent space $T_uM \cong V$ to M at the point u.

As in the proof of Lemma 3.2, for any $u \in \tilde{M}$ there exists a (constant) pseudo-gradient vector field in a local trivialization of the tangent bundle around u. Since a $C^{1,1}$-Finsler manifold is paracompact, see Palais [4; p. 203], and a paracompact $C^{1,1}$-Banach manifold always admits locally Lipschitz partitions of unity, see Palais [4; p. 205 f.], a family of local pseudo-gradient vector fields as above may be patched together to yield a pseudo-gradient vector field $v\colon \tilde{M} \to TM$ just as in the "flat" case $M = V$. Thus we obtain (Palais [4; 3.3. p. 206]):

3.9 Lemma. *Any functional $E \in C^1(M)$ on a $C^{1,1}$-Finsler manifold M admits a pseudo-gradient vector field $v\colon \tilde{M} \to TM$.*

3.10 Remark. Again, if G is a compact Lie group acting (smoothly) through isometries on M and if E is G-invariant, v may be constructed to be G-equivariant in the sense that

$$v\big(g(u)\big) = Dg(u)\, v(u)\ , \quad \forall u \in M,\ g \in G\ .$$

Indeed, we may let

$$\tilde{v}(u) = \int_G (Dg(u))^{-1}\, v\,(g(u))\ dg$$

with respect to an invariant measure dg on G. (Note that in the "flat" case $M = V$, with G acting through linear isometries, the tangent map $Dg(u)\colon T_uM \cong V \to T_{g(u)}M \cong V$ may be identified with the map g itself.)

For $\beta \in \mathbb{R}$, $\delta, \rho > 0$ define K_β, $N_{\beta,\delta}$, $U_{\beta,\delta}$ as in the flat case, using the Finsler structure to define the norm $\|DE(u)\|_u$ and the distance (3.5).

The Palais-Smale condition can now be stated as in Section 2, and Lemma 2.3 remains true for $E \in C^1(M)$ on a $C^{1,1}$-Finsler manifold M. Then the construction in the proof of Theorem 3.4 can be carried over to obtain a local pseudo-gradient flow $\Phi\colon D(\Phi) \subset M \times [0,\infty[\to M$ for E. Note that for Φ to be defined globally on $M \times [0,\infty[$ we also need to assume that M is complete with respect to the metric (3.5). This yields the following result.

3.11 Theorem. *Suppose M is a complete $C^{1,1}$-Finsler manifold and $E \in C^1(M)$ satisfies (P.-S.). Let $\beta \in \mathbb{R}$, $\overline{\varepsilon} > 0$ be given and let N be any neighborhood of K_β. Then there exist a number $\varepsilon \in]0, \overline{\varepsilon}[$ and a continuous 1-parameter family of homeomorphisms $\Phi(\cdot, t)$ of M, $0 \le t < \infty$, with the properties*
(1°) $\Phi(u,t) = u$, if $t = 0$, or $DE(u) = 0$, or $|E(u) - \beta| \ge \overline{\varepsilon}$;
(2°) $E(\Phi(u,t))$ is non-increasing in t for any $u \in M$;
(3°) $\Phi(E_{\beta+\varepsilon} \setminus N, 1) \subset E_{\beta-\varepsilon}$, and $\Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N$.
Moreover, Φ has the semi-group property $\Phi(\cdot, s) \circ \Phi(\cdot, t) = \Phi(\cdot, s+t)$, $\forall s, t \ge 0$. If M admits a compact group of symmetries G and if E is G-invariant, Φ can be constructed to be G-equivariant, that is, such that $\Phi(g(u), t) = g(\Phi(u,t))$ for all $g \in G$, $u \in M, t \ge 0$.

3.12 Remarks. (1°) It suffices to assume that (P.-S.) is satisfied at the level β, see Remark 2.4(2°). In particular, if $N = K_\beta = \emptyset$, condition (P.-S.) may be replaced by the assumption that $N_{\beta,\delta} = \emptyset$ for some $\delta > 0$, see Remark 3.5(3°). (2°) Completeness of M is only needed to ensure that the trajectories of the pseudo-gradient flow Φ are complete in forward direction.

4. The Minimax Principle

The "deformation lemma" Theorem 3.4 is a powerful tool for proving the existence of saddle points of functionals under suitable hypotheses on the topology of the manifold M or the sub-level sets E_α of E.

We proceed to state a very general result in this direction: the generalized minimax principle of Palais [4; p. 210]. Later in this section we sketch some applications of this result.

4.1 Definition. *Let $\Phi: M \times [0, \infty[\to M$ be a semi-flow on a manifold M. A family $\mathcal{F}$ of subsets of M is called (positively) Φ-invariant if $\Phi(F, t) \in \mathcal{F}$ for all $F \in \mathcal{F}$, $t \ge 0$.*

4.2 Theorem. *Suppose M is a complete Finsler manifold of class $C^{1,1}$ and $E \in C^1(M)$ satisfies (P.-S.). Also suppose $\mathcal{F} \subset \mathcal{P}(M)$ is a collection of sets which is invariant with respect to any continuous semi-flow $\Phi: M \times [0, \infty[\to M$ such that $\Phi(\cdot, 0) = id$, $\Phi(\cdot, t)$ is a homeomorphism of M for any $t \ge 0$, and $E(\Phi(u,t))$ is non-increasing in t for any $u \in M$. Then, if*

$$\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u)$$

is finite, β is a critical value of E.

Proof. Assume by contradiction that $\beta \in \mathbb{R}$ is a regular value of E. Choose $\overline{\varepsilon} = 1$, $N = \emptyset$ and let $\varepsilon > 0$, $\Phi: M \times [0, \infty[\to M$ be determined according to Theorem 3.11. By definition of β there exists $F \in \mathcal{F}$ such that

$$\sup_{u\in F} E(u) < \beta + \varepsilon \ ;$$

that is, $F \subset E_{\beta+\varepsilon}$. By property 3.11(3°) of Φ and invariance of $\mathcal{F}$, if we let $F_1 = \Phi(F, 1)$, we have $F_1 \in \mathcal{F}$ and $F_1 \subset E_{\beta-\varepsilon}$; that is,

$$\sup_{u\in F_1} E(u) \leq \beta - \varepsilon \ ,$$

which contradicts the definition of β. □

Of course, it would be sufficient to assume that condition (P.-S.) is satisfied at the level β and that $\mathcal{F}$ is forwardly invariant only with respect to the pseudo-gradient flow.

In the above form, the minimax principle can be most easily applied if E is a functional on a manifold M with a rich topology. But also in the "flat" case $E \in C^1(V)$, V a Banach space, such topological structure may be hidden in the sub-level sets E_γ of E.

4.3 Examples. (Palais, [3; p. 190 f.]) Suppose M is a complete Finsler manifold of class $C^{1,1}$, and $E \in C^1(M)$ satisfies (P.-S.).
(1°) Let $\mathcal{F} = \{M\}$. Then $\mathcal{F}$ is invariant under any semi-flow Φ. Hence, if

$$\beta = \inf_{F\in\mathcal{F}} \sup_{u\in F} E(u) = \sup_{u\in M} E(u)$$

is finite, $\beta = \max_{u\in M} E(u)$ is attained at a critical point of E.
(2°) Let $\mathcal{F} = \{\{u\}; u \in M\}$. Then, if

$$\beta = \inf_{F\in\mathcal{F}} \sup_{u\in F} E(u) = \inf_{u\in M} E(u)$$

is finite, $\beta = \min_{u\in M} E(u)$ is attained at a critical point of E.
(3°) Let X be any topological space, and let $[X, M]$ denote the set of free homotopy classes $[f]$ of continuous maps $f\colon X \to M$. For given $[f] \in [X, M]$ let

$$\mathcal{F} = \{g(X) \ ; \ g \in [f]\} \ .$$

Since $[\Phi \circ f] = [f]$ for any homeomorphism Φ of M homotopic to the identity, the family $\mathcal{F}$ is invariant under such mappings Φ. Hence, if

$$\beta = \inf_{F\in\mathcal{F}} \sup_{u\in F} E(u)$$

is finite, β is a critical value.
(4°) Let $H_k(M)$ denote the k-dimensional homology of M (with arbitrary coefficients). Given a non-trivial element $f \in H_k(M)$, denote by $\mathcal{F}$ the collection of all $F \subset M$ such that f is in the image of

$$H_k(i_F)\colon H_k(F) \to H_k(M) \ ,$$

where $H_k(i_F)$ is the homomorphism induced by the inclusion $i_F: F \hookrightarrow M$. Then $\mathcal{F}$ is invariant under any homeomorphism Φ homotopic to the identity, and by Theorem 4.2, if

$$\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u)$$

is finite, then β is a critical value.

There is a "dual version" of (4°):

(5°) If H^k is any k-dimensional cohomology functor, f a non-trivial element

$$f \in H^k(M), \qquad f \neq 0 \ ,$$

let $\mathcal{F}$ denote the family of subsets $F \subset M$ such that f is not annihilated by the restriction map

$$H^k(i_F): H^k(M) \to H^k(F) \ .$$

Then, if

$$\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u)$$

is finite, β is a critical value.

In the more restricted setting of a functional $E \in C^1(V)$, similar results are valid if M is replaced by any sub-level set E_γ, $\gamma \in \mathbb{R}$. We leave it to the reader to find the analogous variants of (3°), (4°), and (5°). See also Ghoussoub [1].

Closed Geodesics on Spheres

A different construction of a flow-invariant family is at the basis of the next famous result. We assume the notion of geodesic to be familiar from differential geometry. Otherwise the reader may regard (4.2) below as a definition.

4.4 Theorem. *(Birkhoff [1]) On any compact surface S in $\mathbb{R}^3$ which is C^3-diffeomorphic to the standard sphere, there exists a non-constant closed geodesic.*

Proof. Denote $\dot{u} = \frac{d}{dt}u$. Define

$$H^{1,2}\big(\mathbb{R}/2\pi; S\big) = \big\{u \in H^{1,2}_{loc}\left(\mathbb{R}; \mathbb{R}^3\right) \ ; \ u(t) = u(t+2\pi), \\ u(t) \in S \text{ for almost every } t \in \mathbb{R}\big\}$$

the space of closed curves $u: \mathbb{R}/2\pi \to S$ with finite energy

$$E(u) = \frac{1}{2}\int_0^{2\pi} |\dot{u}|^2 \, dt \ .$$

By Hölder's inequality

$$(4.1) \quad |u(s) - u(t)| \leq \int_s^t |\dot{u}| \, d\tau \leq \left(|t-s| \int_s^t |\dot{u}|^2 \, d\tau\right)^{1/2} \leq (2|t-s|E(u))^{1/2} \ ,$$

functions $u \in H^{1,2}(\mathbb{R}/2\pi; S)$ with $E(u) \le \gamma$ will be uniformly Hölder continuous with Hölder exponent $1/2$ and Hölder norm bounded by $\sqrt{2\pi\gamma}$. Hence, if $S \in C^3$, $H^{1,2}(\mathbb{R}/2\pi; S)$ becomes a complete C^2-submanifold of the Hilbert space $H^{1,2}(\mathbb{R}/2\pi; \mathbb{R}^3)$ with tangent space

$$T_u H^{1,2}(\mathbb{R}/2\pi; S) = \{\varphi \in H^{1,2}(\mathbb{R}/2\pi; \mathbb{R}^3) \ ; \ \varphi(t) \in T_{u(t)} S \cong \mathbb{R}^2\}$$

at any closed curve $u \in H^{1,2}(\mathbb{R}/2\pi; S)$; see, for instance, Klingenberg [1; Theorem 1.2.9].

By (4.1), if $E(u)$ is sufficiently small so that the image of u is covered by a single coordinate chart, of course

$$T_u H^{1,2}(\mathbb{R}/2\pi; S) \cong H^{1,2}(\mathbb{R}/2\pi; \mathbb{R}^2) \ .$$

Moreover, E is analytic on $H^{1,2}(\mathbb{R}/2\pi; \mathbb{R}^3)$, hence as smooth as $H^{1,2}(\mathbb{R}/2\pi; S)$ when restricted to that space. At a critical point $u \in C^2$, upon integrating by parts,

$$\int_0^{2\pi} \dot{u}\,\dot{\varphi}\,dt = \int_0^{2\pi} -\ddot{u}\,\varphi\,dt = 0 \ , \quad \forall \varphi \in T_u H^{1,2}(\mathbb{R}/2\pi; S) \ ; \tag{4.2}$$

that is, $\ddot{u}(t) \perp T_{u(t)} S$ for all t, which is equivalent to the assertion that u is a geodesic, parametrized by arc length.

More generally, at a critical point $u \in H^{1,2}(\mathbb{R}/2\pi; S)$, if $n\colon S \to \mathbb{R}^3$ denotes a (C^2-) unit normal vector field on S, for any $\varphi \in H^{1,2}(\mathbb{R}/2\pi; \mathbb{R}^3)$ we have

$$\varphi - n(u)\big(n(u) \cdot \varphi\big) \in T_u H^{1,2}(\mathbb{R}/2\pi; S) \ ,$$

where $\cdot$ is the inner product in $\mathbb{R}^3$. Inserting this into (4.2) and observing that

$$\ddot{u} \cdot n(u) = \big(\dot{u} \cdot n(u)\big)^{\cdot} - \dot{u} \cdot Dn(u)\,\dot{u} = -\dot{u} \cdot Dn(u)\,\dot{u} \tag{4.3}$$

in the distribution sense, we obtain that

$$\ddot{u} + \big(\dot{u} \cdot Dn(u)\,\dot{u}\big) n(u) = 0 \ . \tag{4.4}$$

From (4.4) we now obtain that $u \in H^{2,1}(\mathbb{R}/2\pi) \hookrightarrow C^1(\mathbb{R}/2\pi)$. Hence, by (4.4) again, $u \in C^2(\mathbb{R}/2\pi)$, and our previous discussion shows that closed geodesics on S exactly corresponding to the critical points of E on $H^{1,2}(\mathbb{R}/2\pi; S)$.

Moreover, E satisfies the Palais-Smale condition on $H^{1,2}(\mathbb{R}/2\pi; S)$: If (u_m) is a sequence in $H^{1,2}(\mathbb{R}/2\pi; S)$ such that $E(u_m) \le c < \infty$ and

$$\|DE(u_m)\| = \sup_{\substack{\varphi \in T_{u_m} H^{1,2}(\mathbb{R}/2\pi; S) \\ \|\varphi\|_{1,2} \le 1}} \left| \int_0^{2\pi} \dot{u}_m\,\dot{\varphi}\,dt \right| \to 0 \ ,$$

then (u_m) contains a strongly convergent subsequence.

Proof of (P.-S.). Since $E(u_m) \le c$ uniformly, by (4.1) the sequence (u_m) is equi-continuous. But S is compact, in particular bounded; hence (u_m) is equi-bounded. Thus, by Arzéla-Ascoli's theorem we may assume that $u_m \to u$ uniformly and weakly in $H^{1,2}(\mathbb{R}/2\pi;\mathbb{R}^3)$. It follows that $u \in H^{1,2}(\mathbb{R}/2\pi;S)$.

Via the unit normal vector field n we can define the projection $\pi_v\colon H^{1,2}(\mathbb{R}/2\pi;\mathbb{R}^3) \to T_vH^{1,2}(\mathbb{R}/2\pi;S)$ by letting

$$\varphi \mapsto \pi_v\varphi(t) = \varphi(t) - n\,(v(t))\,\big(n\,(v(t))\cdot\varphi(t)\big),$$

as above. In particular, we have

$$\varphi_m := \pi_{u_m}(u_m - u) \in T_{u_m}H^{1,2}(\mathbb{R}/2\pi;S)\ .$$

Note that, since $n \in C^2$ and since (u_m) is bounded in $H^{1,2}(\mathbb{R}/2\pi;S)$, the sequence (φ_m) is bounded in $H^{1,2}$. Thus, $\langle\varphi_m, DE(u_m)\rangle \to 0$. Moreover, since $u_m \rightharpoonup u$ weakly in $H^{1,2}$ and uniformly, it follows that also $\varphi_m \rightharpoonup 0$ weakly in $H^{1,2}$ and uniformly. Consequently, we have

$$\begin{aligned}
o(1) = \langle\varphi_m, DE(u_m)\rangle &= \int_0^{2\pi} \dot u_m\dot\varphi_m\,dt\\
&= \int_0^{2\pi} (\dot u_m - \dot u)\dot\varphi_m\,dt + o(1)\\
&= \int_0^{2\pi} |\dot u_m - \dot u|^2 - (\dot u_m - \dot u)\cdot\frac{d}{dt}\big[n(u_m)\big(n(u_m)\cdot(u_m-u)\big)\big]\,dt + o(1)\\
&= \int_0^{2\pi} |\dot u_m - \dot u|^2 - \big((\dot u_m - \dot u)\cdot n(u_m)\big)\big(n(u_m)\cdot(\dot u_m - \dot u)\big)\,dt + o(1)\ ,
\end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$. But

$$n(u)\cdot\dot u = 0 = n(u_m)\cdot\dot u_m$$

almost everywhere. Hence

$$\begin{aligned}
\int_0^{2\pi} |n(u_m)(\dot u_m - \dot u)|^2\,dt &= \int_0^{2\pi} |\,(n(u_m) - n(u))\,\dot u|^2\,dt\\
&\le 2\|n(u_m) - n(u)\|_\infty^2 E(u) \to 0 \quad (m\to\infty)\ ,
\end{aligned}$$

and $u_m \to u$ strongly. □

In order to construct a flow-invariant family $\mathcal{F}$ we now proceed as follows:

By assumption, there exists a C^3-diffeomorphism $\Psi\colon S \to S^2$ from S onto the standard sphere $S^2 \subset \mathbb{R}^3$. Let $p\colon[-\frac{\pi}{2},\frac{\pi}{2}] \to H^{1,2}(\mathbb{R}/2\pi;S)$ be a 1-parameter family of closed curves $u = p(\theta)$ on S, such that $p(\pm\frac{\pi}{2}) \equiv p_\pm$ are constant "curves" in S. Via Ψ we associate with p a map $\tilde p \in C^0(S^2;S^2)$ by representing $S^2 \cong [-\frac{\pi}{2},\frac{\pi}{2}]\times\mathbb{R}/2\pi$ in polar coordinates (θ,ϕ), with $\{-\frac{\pi}{2}\}\times\mathbb{R}/2\pi$, respectively $\{\frac{\pi}{2}\}\times\mathbb{R}/2\pi$, collapsed to points. Then we let

$$\tilde{p}(\theta,\phi) = \Psi\left(p(\theta)(\phi)\right) \ .$$

Consider now the collection

$$P = \{p \in C^0\left([-\frac{\pi}{2},\frac{\pi}{2}];\ H^{1,2}(\mathbb{R}/2\pi;S)\right)\ ;\ p(\pm\frac{\pi}{2}) \equiv const. \in S\}$$

and let

$$\mathcal{F} = \{p \in P\ ;\ \tilde{p} \text{ is homotopic to } id\big|_{S^2}\} \ .$$

Choosing for $p(\theta)$ the pre-image under Ψ of a family of equilateral circles covering S^2, we find that $\mathcal{F} \neq \emptyset$. Also note that the map $P \ni p \mapsto \tilde{p} \in C^0(S^2;S^2)$ is continuous. Hence $\mathcal{F}$ is Φ-invariant under any homeomorphism Φ of $H^{1,2}(\mathbb{R}/2\pi;S)$ homotopic to the identity, and which maps constant maps to constant maps. Note that, in particular, any Φ which does not increase E will have this latter property.

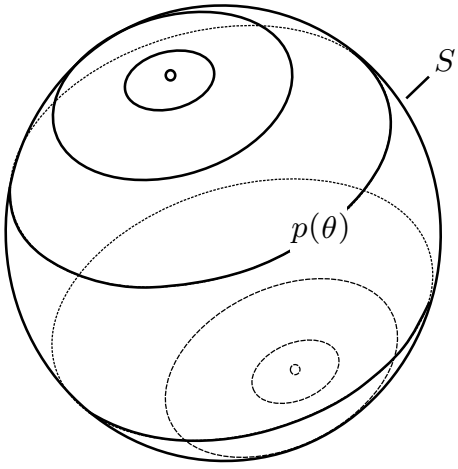

Fig. 4.1. An admissible comparison path $p \in P$

Finally, by Theorem 4.2

$$\beta = \inf_{p\in\mathcal{F}} \sup_{u\in p} E(u)$$

is critical.

This almost completes the proof of Theorem 4.4. However, it remains to rule out the possibility that $\beta = 0$: the energy of trivial (constant) "closed geodesics".

4.5 Lemma. $\beta > 0$.

Proof. There exists $\delta > 0$ such that for any x at distance $dist(x, S) \le \delta$ from S there is a unique nearest neighbor $\pi(x) \in S$, characterized by

$$|\pi(x) - x| = \inf_{y \in S} |x - y| \ ,$$

and $\pi(x)$ depends continuously on x. Moreover, π is C^2 if S is of class C^3. By (4.1) there exists $\gamma > 0$ such that for $u \in H^{1,2}(\mathbb{R}/2\pi; S)$ with $E(u) \le \gamma$ there holds

$$\operatorname{diam}(u) = \sup_{0 \le \phi, \phi' \le 2\pi} |u(\phi) - u(\phi')| < \delta \ . \tag{4.5}$$

Now suppose $\beta < \gamma$, and let $p \in \mathcal{F}$ be such that $E(u) \le \gamma$ for any curve $u = p(\theta) \in p$. By (4.5), if we fix $\phi_0 \in [0, 2\pi]$, we can continuously contract any such curve u to the constant curve $u(\phi_0)$ in the δ-neighborhood of S by letting

$$u_s(\phi) = (1 - s)u(\phi) + s\, u(\phi_0),\ 0 \le s \le 1 \ .$$

Composing with π, we obtain a homotopy

$$p_s(\theta, \phi) = \pi\big((1 - s)p(\theta)(\phi) + sp(\theta)(\phi_0)\big),\ 0 \le s \le 1,$$

between $p = p_0$ and a path $p_1 \in P$ consisting entirely of constant loops $p_1(\theta) \equiv p(\theta)(\phi_0)$ for all θ. Composing p_s with $\Psi \colon S \to S^2$, we also obtain a homotopy of $\tilde{p} = \tilde{p}_0 \sim id\big|_{S^2}$ to the map $\tilde{p}_1$, given by $\tilde{p}_1(\theta, \phi) = \Psi\,(p(\theta)(\phi_0))$. But, letting

$$\tilde{p}_{1,r}(\theta, \phi) = \tilde{p}_1(r\theta, \phi) = \Psi\big(p(r\theta)(\phi_0)\big) \ ,\ 0 \le r \le 1 \ ,$$

we see that $\tilde{p}_1$ – and hence also $\tilde{p}$ – is homotopic to a constant map, contrary to our choice of $p \in \mathcal{F}$. □

4.6 Notes. Birkhoff's result of 1917 and a later extension to spheres of arbitrary dimension (Birkhoff [2]) mark the beginning of the calculus of variations in the large. A major advance then came with the celebrated work of Lusternik-Schnirelmann [1] in 1929 who – by variational techniques – established the existence of three geometrically distinct closed geodesics free of self-intersections on any compact surface of genus 0. (Detailed proofs were published by Lusternik [1] in 1947.)

For recent developments in the theory of closed geodesics, see, for instance, Klingenberg [1], Bangert [1], and Franks [1].

5. Index Theory

In most cases the topology of the space M where a functional E is defined will be rather poor. However, if E is invariant under a compact group G acting on M, this may change drastically if we can pass to the quotient M/G with respect to the symmetry group. Often this space will have a richer topological structure which we may hope to exploit in order to obtain multiple critical points.

However, in general the group G will not act freely on M and the quotient space will be singular, in particular, it will no longer be a manifold. Therefore the results outlined in the preceding sections cannot be applied.

A nice way around this difficulty is to consider flow-invariant families $\mathcal{F}$ in Theorem 4.2 which are also invariant under the group action. Since by Remark 3.5.(4°) we may choose our pseudo-gradient flows Φ to be equivariant if E is, this approach is promising. Moreover, at least for special kinds of group actions G, the topological complexity of the elements of such equivariant families can be easily measured or estimated in terms of an "index" which then may be used to distinguish different critical points.

Krasnoselskii Genus

The concept of an index theory is most easily explained for an even functional E on some Banach space V, with symmetry group $G = \mathbb{Z}_2 = \{id, -id\}$. Define

$$\mathcal{A} = \{A \subset V\,;\ A \text{ closed},\ A = -A\}$$

to be the class of closed symmetric subsets of V.

5.1 Definition. For $A \in \mathcal{A},\ A \neq \emptyset$, following Coffman [1], let

$$\gamma(A) = \begin{cases} \inf\{m\ ;\ \exists h \in C^0(A; \mathbb{R}^m \setminus \{0\}),\ h(-u) = -h(u)\} \\ \infty, \qquad \text{if } \{..\} = \emptyset, \text{ in particular, if } A \ni 0\ , \end{cases}$$

and define $\gamma(\emptyset) = 0$. Note that for any $A \in \mathcal{A}$, by the Tietze extension theorem, any odd map $h \in C^0(A; \mathbb{R}^m)$ may be extended to a map $\tilde{h} \in C^0(V; \mathbb{R}^m)$. Letting $h(u) = \frac{1}{2}\big(\tilde{h}(u) - \tilde{h}(-u)\big)$, the extension can be chosen to be symmetric.

$\gamma(A)$ is called the Krasnoselskii genus of A. (The equivalence of Coffman's definition above with Krasnoselskii's [1] original definition – see also Krasnoselskii-Zabreiko [1; p. 385 ff.] – was established by Rabinowitz [1; Lemma 3.6].) A notion of coindex with related properties was introduced by Connor-Floyd [1].

The notion of genus generalizes the notion of dimension of a linear space:

5.2 Proposition. *For any bounded symmetric neighborhood Ω of the origin in $\mathbb{R}^m$ there holds: $\gamma(\partial\Omega) = m$.*

Proof. Trivially, $\gamma(\partial\Omega) \leq m$. (Choose $h = id$.) Let $\gamma(\partial\Omega) = k$ and let $h \in C^0(\mathbb{R}^m; \mathbb{R}^k)$ be an odd map such that $h(\partial\Omega) \not\ni 0$. We may consider $\mathbb{R}^k \subset \mathbb{R}^m$. But then the topological degree of $h: \mathbb{R}^m \to \mathbb{R}^k \subset \mathbb{R}^m$ on Ω with respect to 0 is well-defined (see Deimling [1; Definition 1.2.3]). In fact, since h is odd, by the Borsuk-Ulam theorem (see Deimling [1; Theorem 1.4.1]) we have

$$\deg(h, \Omega, 0) = 1 \ .$$

Hence by continuity of the degree also

$$\deg(h, \Omega, y) = 1 \neq 0$$

for $y \in \mathbb{R}^m$ close to 0 and thus, by the solution property of the degree, h covers a neighborhood of the origin in $\mathbb{R}^m$; see Deimling [1; Theorem 1.3.1]. But then $k = m$, as claimed. □

Proposition 5.2 has a converse:

5.3 Proposition. *Suppose $A \subset V$ is a compact symmetric subset of a Hilbert space V with inner product $(\cdot, \cdot)_V$, and suppose $\gamma(A) = m < \infty$. Then A contains at least m mutually orthogonal vectors u_k, $1 \leq k \leq m$, $(u_k, u_l)_V = 0$ $(k \neq l)$.*

Proof. Let $u_1, \ldots, u_l$ be a maximal set of mutually orthogonal vectors in A, and denote $W = \text{span}\,\{u_1, \ldots, u_l\} \cong \mathbb{R}^l$, $\pi: V \to W$ orthogonal projection onto W. Then $\pi(A) \not\ni 0$, and π defines an odd continuous map $h = \pi|_A: A \to \mathbb{R}^l \setminus \{0\}$. By definition of $\gamma(A) = m$ we conclude that $l \geq m$, as claimed. □

Moreover, the genus has the following properties:

5.4 Proposition. *Let $A, A_1, A_2 \in \mathcal{A}$, $h \in C^0(V; V)$ an odd map. Then the following hold:*
(1°) $\gamma(A) \geq 0$, $\gamma(A) = 0 \quad \Leftrightarrow \quad A = \emptyset$.
(2°) $A_1 \subset A_2 \Rightarrow \gamma(A_1) \leq \gamma(A_2)$.
(3°) $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$.
(4°) $\gamma(A) \leq \gamma\left(\overline{h(A)}\right)$.
(5°) If $A \in \mathcal{A}$ is compact and $0 \notin A$, then $\gamma(A) < \infty$ and there is a neighborhood N of A in V such that $\overline{N} \in \mathcal{A}$ and $\gamma(A) = \gamma(\overline{N})$.
That is, γ is a definite, monotone, sub-additive, supervariant and "continuous" map $\gamma: \mathcal{A} \to \mathbb{N}_0 \cup \{\infty\}$.

Proof. (1°) follows by definition.

(2°) If $\gamma(A_2) = \infty$ we are done. Otherwise, suppose $\gamma(A_2) = m$. By definition there exists $h \in C^0(A_2; \mathbb{R}^m \setminus \{0\})$, $h(-u) = -h(u)$. Restricting h to A_1 yields an odd map $h|_{A_1} \in C^0(A_1; \mathbb{R}^m \setminus \{0\})$, whence $\gamma(A_1) \le \gamma(A_2)$.
(3°) Again we may suppose that both $\gamma(A_1) = m_1$, $\gamma(A_2) = m_2$ are finite, and we may let h_1, h_2 be odd maps $h_i \in C^0(A_i; \mathbb{R}^{m_i} \setminus \{0\})$, $i = 1, 2$, as in the definition of the genus. As noted above, we may extend h_1, h_2 to odd maps $h_i \in C^0(V; \mathbb{R}^{m_i})$, $i = 1, 2$. But then letting $h(u) = (h_1(u), h_2(u))$ defines an odd map $h \in C^0(V; \mathbb{R}^{m_1+m_2})$ which does not vanish for $u \in A_1 \cup A_2$, and claim (3°) follows.
(4°) Any odd map $\tilde{h} \in C^0\left(\overline{h(A)}; \mathbb{R}^m \setminus \{0\}\right)$ induces an odd map $\tilde{h} \circ h \in C^0(A; \mathbb{R}^m \setminus \{0\})$, and (4°) is immediate.
(5°) If A is compact and $0 \notin A$ there is $\rho > 0$ such that $A \cap B_\rho(0) = \emptyset$. The cover $\left\{\tilde{B}_\rho(u) = B_\rho(u) \cup B_\rho(-u)\right\}_{u \in A}$ of A admits a finite sub-cover $\{\tilde{B}_\rho(u_1), \dots, \tilde{B}_\rho(u_m)\}$. Let $\{\varphi_j\}_{1 \le j \le m}$ be a partition of unity on A subordinate to $\{\tilde{B}_\rho(u_j)\}_{1 \le j \le m}$; that is, let $\varphi_j \in C^0\left(\tilde{B}_\rho(u_j)\right)$ with support in $\tilde{B}_\rho(u_j)$ satisfy $0 \le \varphi_j \le 1$, $\sum_{j=1}^m \varphi_j(u) = 1$ for all $u \in A$. Replacing φ_j by $\overline{\varphi}_j(u) = \frac{1}{2}\big(\varphi_j(u) + \varphi_j(-u)\big)$, if necessary, we may assume that φ_j is even, $1 \le j \le m$. By choice of ρ, for any j the neighborhoods $B_\rho(u_j)$, $B_\rho(-u_j)$ are disjoint.

Hence the map $h\colon V \to \mathbb{R}^m$ with jth component

$$h_j(u) = \begin{cases} \varphi_j(u), & \text{if } u \in B_\rho(u_j)\ , \\ -\varphi_j(u), & \text{if } u \in B_\rho(-u_j)\ , \\ 0, & \text{else}\ , \end{cases}$$

is continuous, odd, and does not vanish on A. This shows that $\gamma(A) < \infty$.

Finally, assume that A is compact, $0 \notin A$, $\gamma(A) = m < \infty$ and let $h \in C^0(A; \mathbb{R}^m \setminus \{0\})$ be as in the definition of $\gamma(A)$. We may assume $h \in C^0(V; \mathbb{R}^m)$. Moreover, A being compact, also $h(A)$ is compact, and there exists a symmetric open neighborhood $\tilde{N}$ of $h(A)$ whose closure is compactly contained in $\mathbb{R}^m \setminus \{0\}$. Choosing $N = h^{-1}(\tilde{N})$, by construction $h(\overline{N}) \not\ni 0$ and $\gamma(\overline{N}) \le m$. On the other hand, $A \subset N$. Hence $\gamma(\overline{N}) = \gamma(A)$ by monotonicity of γ, property (2°). □

5.5 Observation. It is easy to see that if A is a finite collection of antipodal pairs $u_i, -u_i$ ($u_i \neq 0$), then $\gamma(A) = 1$.

Minimax Principles for Even Functionals

Suppose E is a functional of class C^1 on a closed symmetric $C^{1,1}$-submanifold M of a Banach space V and satisfies (P.-S.). Moreover, suppose that E is even, that is, $E(u) = E(-u)$ for all u. Also let $\mathcal{A}$ be as above. Then for any $k \le \gamma(M) \le \infty$ by Proposition 5.4(4°) the family

$$\mathcal{F}_k = \{A \in \mathcal{A}\ ;\ A \subset M,\ \gamma(A) \geq k\}$$

is invariant under any odd and continuous map and non-empty. Hence, analogous to Theorem 4.2, for any $k \leq \gamma(M)$, if

$$\beta_k = \inf_{A \in \mathcal{F}_k} \sup_{u \in A} E(u)$$

is finite, then β_k is a critical value of E; see Theorem 5.7 below.

To see what happens when β_k and β_{k+1} coincide for some k, it is instructive to compare this result with the well-known Courant-Fischer minimax principle for linear eigenvalue problems. Recall that on $\mathbb{R}^n$ with scalar product $(\cdot,\cdot)$ the k-th eigenvalue of a symmetric linear map $K\colon \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\lambda_k = \min_{\substack{V' \subset V,\\ \dim V' = k}} \max_{\substack{u \in V'\\ \|u\|=1}} (Ku, u)\ .$$

Translated into the above setting, we may likewise determine λ_k by considering the functional

$$E(u) = (Ku, u)$$

on the unit sphere $M = S^{n-1}$ and computing β_k as above. Trivially, E satisfies (P.-S.); moreover, it is easy to see that $\beta_k = \lambda_k$ for all k. (By Proposition 5.2 the inequality $\beta_k \leq \lambda_k$ is immediate. The reverse inequality follows by Proposition 5.3 and linearity of K.)

In the linear case, now, it is clear that if successive eigenvalues $\lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+l-1} = \lambda$ coincide, then K has an l-dimensional eigenspace of eigenvectors $u \in V$ satisfying $Ku = \lambda u$. Is there a similar result in the nonlinear setting? Actually, there is. For this we again assume that $E \in C^1(M)$ is an even functional on a closed, symmetric $C^{1,1}$-submanifold $M \subset V \setminus \{0\}$, satisfying (P.-S.). Let $\beta_k,\ k \leq \gamma(M)$, be defined as above.

5.6 Lemma. *Suppose for some k, l there holds*

$$-\infty < \beta_k = \beta_{k+1} = \ldots = \beta_{k+l-1} = \beta < \infty\ .$$

Then $\gamma(K_\beta) \geq l$. By Observation 5.5, in particular, if $l > 1$, K_β is infinite.

Proof. By (P.-S.) the set K_β is compact and symmetric. Hence $\gamma(K_\beta)$ is well-defined and by Proposition 5.4(5°) there exists a symmetric neighborhood N of K_β in M such that $\gamma(\overline{N}) = \gamma(K_\beta)$. For $\overline{\varepsilon} = 1,\ N$, and β as above let $\varepsilon > 0$ and Φ be determined according to Theorem 3.11. We may assume Φ is odd. Choose $A \subset M$ such that $\gamma(A) \geq k + l - 1$ and $E(u) < \beta + \varepsilon$ for $u \in A$.

Let $\overline{\Phi(A, 1)} = \tilde{A} \in \mathcal{A}$. By property (3°) of Φ in Theorem 3.11

$$\tilde{A} \subset \overline{(E_{\beta-\varepsilon} \cup N)}\ .$$

Moreover, by definition of $\beta = \beta_k$ it follows that

$$\gamma\left(\overline{E_{\beta-\varepsilon}}\right) < k\ .$$

Thus, by Proposition 5.4(2°)–5.4(4°)we have

$$\begin{aligned}\gamma(\overline{N}) &\geq \gamma\left(\overline{E_{\beta-\varepsilon} \cup N}\right) - \gamma\left(\overline{E_{\beta-\varepsilon}}\right) \\ &> \gamma(\tilde{A}) - k \;\geq\; \gamma(A) - k \\ &\geq k + l - 1 - k = l - 1\ ;\end{aligned}$$

that is, $\gamma(\overline{N}) = \gamma(K_\beta) \geq l$, as claimed. □

In consequence, we note

5.7 Theorem. *Suppose $E \in C^1(M)$ is an even functional on a complete symmetric $C^{1,1}$-manifold $M \subset V \setminus \{0\}$ in some Banach space V. Also suppose E satisfies (P.-S.) and is bounded from below on M. Let $\hat{\gamma}(M) = \sup\{\gamma(K)\ ;\ K \subset M$ compact and symmetric$\}$. Then the functional E possesses at least $\hat{\gamma}(M) \leq \infty$ pairs of critical points.*

Remarks. Note that the definition of $\hat{\gamma}(M)$ assures that for $k \leq \hat{\gamma}(M)$ the numbers β_k are finite.

Completeness of M can be replaced by the assumption that the flow defined by any pseudo-gradient vector field on M exists for all positive time.

Applications to Semilinear Elliptic Problems

As a particular case, Theorem 5.7 includes the following classical result of Lusternik-Schnirelmann [2]: Any even function $E \in C^1(\mathbb{R}^n)$ admits at least n distinct pairs of critical points when restricted to S^{n-1}. In infinite dimensions, Theorem 5.7 and suitable variants of it have been applied to the solution of nonlinear partial differential equations and nonlinear eigenvalue problems with a $\mathbb{Z}_2$-symmetry. See for instance Amann [1], Clark [1], Coffman [1], Hempel [1], Rabinowitz [7], Thews [1], and the surveys and lecture notes by Browder [2], Rabinowitz [7], [11].

Here we present only a simple example of this kind for which we return to the setting of problem

$$(I.2.1) \qquad -\Delta u + \lambda u = |u|^{p-2}u \qquad \text{in } \Omega\ ,$$

$$(I.2.3) \qquad u = 0 \qquad \text{on } \partial\Omega\ ,$$

considered earlier. This time, however, we also admit solutions of varying sign.

5.8 Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $p > 2$; if $n \geq 3$ we assume in addition $p < \frac{2n}{n-2}$. Then for any $\lambda \geq 0$ problem (I.2.1), (I.2.3) admits infinitely many distinct pairs of solutions.*

Proof. By Theorem 5.7 the even functional

$$E(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda |u|^2\right)\, dx$$

admits infinitely many distinct pairs of critical points on the sphere $S = \{u \in H_0^{1,2}\ ;\ \|u\|_{L^p} = 1\}$, for any $\lambda \geq 0$. Scaling suitably, we obtain infinitely many distinct pairs of solutions for (I.2.1), (I.2.3). □

General Index Theories

The concept of index can be generalized. Our presentation is based on Rabinowitz [11]. Related material can also be found in the recent monograph by Bartsch [1]. Suppose M is a complete $C^{1,1}$-Finsler manifold with a compact group action G. Let

$$\mathcal{A} = \{A \subset M\ ;\ A \text{ is closed, } g(A) = A \text{ for all } g \in G\}$$

be the set of G-invariant subsets of M, and let

$$\Gamma = \{h \in C^0(M; M)\ ;\ h \circ g = g \circ h \text{ for all } g \in G\}$$

be the class of G-equivariant mappings of M. (Since our main objective is that the flow $\Phi(\,\cdot\,, t)$ constructed in Theorem 3.11 be in Γ, we might also restrict Γ to the class of G-equivariant homeomorphisms of M.) Finally, if $G \neq \{id\}$, denote

$$\text{Fix } G = \left\{u \in M\ ;\ gu = u \text{ for all } g \in G\right\}$$

the set of fixed points of G.

5.9 Definition. *An index for $(G, \mathcal{A}, \Gamma)$ is a mapping $i\colon \mathcal{A} \to \mathbb{N}_0 \cup \{\infty\}$ such that for all $A, B \in \mathcal{A}$, $h \in \Gamma$ there holds*
(1°) (definiteness:) $i(A) \geq 0$, $i(A) = 0 \iff A = \emptyset$.
(2°) (monotonicity:) $A \subset B \Rightarrow i(A) \leq i(B)$.
(3°) (sub-additivity:) $i(A \cup B) \leq i(A) + i(B)$.
(4°) (supervariance:) $i(A) \leq i\left(\overline{h(A)}\right)$.
(5°) (continuity:) If A is compact and $A \cap Fix\ G = \emptyset$, then $i(A) < \infty$ and there is a G-invariant neighborhood N of A such that $i(\overline{N}) = i(A)$.
(6°) (normalization): If $u \notin Fix\, G$, then $i\left(\bigcup_{g\in G} gu\right) = 1$.

5.10 Remarks and examples. (1°) If $A \in \mathcal{A}$ and $A \cap Fix\, G \neq \emptyset$ then $i(A) = \sup_{B\in\mathcal{A}} i(B)$; indeed, by monotonicity, for $u_0 \in A \cap Fix\, G$ there holds $i(\{u_o\}) \leq i(A) \leq \sup_{B\in\mathcal{A}} i(B)$. On the other hand, for any $B \in \mathcal{A}$ the map $h\colon B \to \{u_0\}$, given by $h(u) = u_0$ for all $u \in B$, is continuous and equivariant, whence $i(B) \leq i(\{u_0\})$ by supervariance of the index. Hence, in general nothing will be lost if we define $i(A) = \infty$ for $A \in \mathcal{A}$ such that $A \cap Fix\, G \neq \emptyset$.

(2°) By Example 5.1 the Krasnoselskii genus γ is an index for $G = \{id, -id\}$, the class $\mathcal{A}$ of closed, symmetric subsets, and Γ the family of odd, continuous maps.

Analogous to Theorem 5.7 we have the following general existence result for variational problems that admit an index theory.

5.11 Theorem. *Let $E \in C^1(M)$ be a functional on a complete $C^{1,1}$-Finsler manifold M and suppose E is bounded from below and satisfies (P.-S.). Suppose G is a compact group acting on M without fixed points and let $\mathcal{A}$ be the set of closed G-invariant subsets of M, Γ be the group of G-equivariant homeomorphisms of M. Suppose i is an index for $(G, \mathcal{A}, \Gamma)$, and let*

$$\hat{i}(M) = \sup\{i(K)\ ;\ K \subset M \text{ is compact and } G - \text{invariant}\} \leq \infty\ .$$

Then E admits at least $\hat{i}(M)$ critical points which are distinct modulo G.

The proof is the same as that of Theorem 5.7 and Lemma 5.6. Again note that completeness of M can be replaced by the assumption that any pseudo-gradient flow on M is complete in forward time.

Lusternik-Schnirelman Category

The concept of category was introduced by Lusternik-Schnirelmann [2]. This notion, in fact, is the first example of an index theory (in the above sense) in the mathematical literature.

5.12 Definition. *Let M be a topological space and consider a closed subset $A \subset M$. We say that A has category k relative to M ($\mathrm{cat}_M(A) = k$), if A is covered by k closed sets A_j, $1 \leq j \leq k$, which are contractible in M, and if k is minimal with this property. If no such finite covering exists, we let $\mathrm{cat}_M(A) = \infty$. Moreover, we define $\mathrm{cat}_M(\emptyset) = 0$.*

This notion fits in the frame of Definition 5.9 if we let

$$G = \{id\}$$
$$\mathcal{A} = \{A \subset M\ ;\ A \text{ closed }\},$$
$$\Gamma = \{h \in C^0(M; M)\ ;\ h \text{ is a homeomorphism }\}.$$

Then we have

5.13 Proposition. *cat_M is an index for $(G, \mathcal{A}, \Gamma)$.*

Proof. (1°)–(3°) of Definition 5.9 are immediate. (4°) is also clear, since a homeomorphism h preserves the topological properties of any sets A_j covering A. (5°) Any open cover of a compact set A by open sets O whose closure is contractible has a finite subcover $\{O_j\ ,\ 1 \leq j \leq k\}$. Set $N = \bigcup O_j$. (6°) is obvious. □

5.14 Categories of some standard sets. (1°) If $M = T^m = \mathbb{R}^m/\mathbb{Z}^m$ is the m-torus, then $\mathrm{cat}_{T^m}(T^m) = m+1$, see Lusternik-Schnirelman [2] or Schwartz [2; Lemma 5.15, p. 161]. Thus, any functional $E \in C^1(T^m)$ possesses at least $m+1$ distinct critical points. In particular, if $m = 2$, any C^1-functional on the standard torus, besides an absolute minimum and maximum, must possess at least one additional critical point.
(2°) For the m-sphere $S^m \subset \mathbb{R}^{m+1}$ we have $\mathrm{cat}_{S^m}(S^m) = 2$. (Take A_1, A_2 slightly overlapping northern and southern hemispheres.)
(3°) For the unit sphere S in an infinite dimensional Banach space we have $\mathrm{cat}_S(S) = 1$. (S is contractible in itself.)
(4°) For real or complex m-dimensional projective space P^m we have $\mathrm{cat}_{P^m}(P^m) = m+1$ $(m \leq \infty)$.

Since real projective $P^m = S^m/\mathbb{Z}_2$, we may ask whether, in the presence of a $\mathbb{Z}_2$-symmetry $u \to -u$, the category and Krasnoselskii genus of symmetric sets are always related as in the above example 5.14(4°). This is indeed the case, see (Rabinowitz [1; Theorem 3.7]):

5.15 Proposition. *Suppose $A \subset \mathbb{R}^m \setminus \{0\}$ is compact and symmetric, and let $\tilde{A} = A/\mathbb{Z}_2$ with antipodal points collapsed. Then $\gamma(A) = \mathrm{cat}_{\mathbb{R}^m\setminus\{0\}/\mathbb{Z}_2}(\tilde{A})$.*

Using the notion of category, results in the spirit of Theorem 5.8 have been established by Browder [1], [3] and Schwartz [1], for example.

With index theories offering a very convenient means to characterize different critical points of functionals possessing certain symmetries, it is not surprising that, besides the classical examples treated above, a variety of other index theories have been developed. See the papers by Fadell-Husseini [1], Fadell-Husseini-Rabinowitz [1], Fadell-Rabinowitz [1] on cohomological index theories – a very early paper in this regard is Yang [1]. Relative or pseudo-indices were introduced by Benci [3] and Bartolo-Benci-Fortunato [1].

For our final model problem in this section it will suffice to consider the S^1-index of Benci [2] as another particular case.

A Geometrical S^1-Index

If M is a complete $C^{1,1}$-Finsler manifold with an S^1-action (in particular, if M is a complex Hilbert space with $S^1 = \{e^{i\phi} \,;\, 0 \leq \phi \leq 2\pi\}$ acting through scalar multiplication) we may define an index for this action as follows; see Benci [2].

5.16 Definition. Let $\mathcal{A}$ be the family of closed, S^1-invariant subsets of M, and Γ the family of S^1-equivariant maps (or homeomorphisms). For $A \neq \emptyset$, define

$$\tau(A) = \begin{cases} \inf\big\{m \,;\, \exists h \in C^0\big(A; \mathbb{C}^m \setminus \{0\}\big),\ l \in \mathbb{N}: \\ \qquad h \circ g = g^l \circ h \text{ for all } g \in S^1\big\}, & \text{if } \{\ldots\} \neq \emptyset, \\ \infty, & \text{if } \{\ldots\} = \emptyset, \end{cases}$$

and let $\tau(\emptyset) = 0$. (Note the similarity with the Krasnoselskii index γ.) Here, S^1 acts on $\mathbb{C}^m$ by component-wise complex multiplication.

5.17 Proposition. *τ is an index for $(S^1, \mathcal{A}, \Gamma)$.*

Proof. It is easy to see that τ satisfies properties (1°), (2°), and (4°) of Definition 5.9. To see (3°) we may assume that $\tau(A_i) = m_i < \infty,\ i = 1, 2$, and we may choose h_i, l_i as in the definition of τ such that $h_i \in C^0(A_i; \mathbb{C}^{m_i} \setminus \{0\})$ satisfies $h_i \circ g = g^{l_i} \circ h_i$ for all $g \in S^1,\ i = 1, 2$.

Extending h_i to M and averaging

$$\tilde{h}_i(u) = \int_{S^1} g^{-l_i} h_i(gu)\, dg$$

with respect to an invariant measure (arc-length) on S^1, we may assume that $h_i \in C^0(M; \mathbb{C}^m),\ i = 1, 2$. But then the map

$$h(u) = \Big(\big(h_1(u)\big)^{l_2}, \big(h_2(u)\big)^{l_1} \Big)\ ,$$

where for $(z_1, \dots, z_m) \in \mathbb{C}^m,\ l \in \mathbb{N}$ we let $(z_1, \dots, z_m)^l := (z_1^l, \dots, z_m^l)$, defines a map

$$h \in C^0\big(A_1 \cup A_2\ ;\ \mathbb{C}^{m_1+m_2} \setminus \{0\}\big)$$

such that $h \circ g = \Big(\big(g^{l_1} \circ h_1\big)^{l_2}, \big(g^{l_2} \circ h_2\big)^{l_1} \Big) = g^l \circ h$ for all $g \in S^1$ with $l = l_1 l_2$.

To see (6°), for an element $u_0 \notin Fix\,(S^1)$ let

$$G_0 = \{g \in S^1\ ;\ gu_0 = u_0\}$$

be the subgroup of S^1 fixing u_0. Since $u_0 \notin Fix\,(S^1)$, G_0 is discrete, hence represented by

$$G_0 = \big\{e^{2\pi i k/l}\ ;\ 0 \le k < l\big\}$$

for some $l \in \mathbb{N}$.

For $u = gu_0$ now let $h(u) = g^l \in S^1 \subset \mathbb{C} \setminus \{0\}$. Then h is well-defined and continuous along the S^1-orbit $S^1 u_0 = \{gu_0\ ;\ g \in S^1\}$ of u_0. Extending h equivariantly, we see that $\tau(S^1 u_0) = 1$, which proves (6°).

Finally, to see (5°), suppose A is S^1-invariant, compact, and $A \cap \mathrm{Fix}(S^1) = \emptyset$. For any $u_0 \in A$ let h be constructed as in the proof of (6°) and let $O(u_0)$ be an S^1-invariant neighborhood of $S^1 u_0$ such that $h\big(\overline{O(u_0)}\big) \not\ni 0$.

By compactness of A, finitely many such neighborhoods $\{O(u_i)\}_{1 \le i \le m}$ cover A, whence $\tau(A) < \infty$ by sub-additivity. The remainder of the proof of (5°) is the same as in Proposition 5.4. □

As in the case of the Krasnoselskii genus, Proposition 5.2, the S^1-index may be interpreted as a generalized dimension of a closed S^1-invariant set; see Benci [2; Proposition 2.6]. However, we will not pursue this.

Instead, we observe that in the case of a free S^1-action on a manifold M, a simpler variant of Benci's S^1-index can be defined as follows. (Recall that a group G acts freely on a manifold M if only the identity element in G fixes points in M.)

5.16′ Definition. Suppose S^1 acts freely on a manifold M. Let $\mathcal{A}$ be the family of closed, S^1-invariant subsets of M, and Γ the family of S^1-equivariant maps (or homeomorphisms). For $A \neq \emptyset$, define

$$\tilde{\tau}(A) = \begin{cases} \inf\{m \ ; \ \exists h \in C^0(A; \mathbb{C}^m \setminus \{0\}) : \\ \qquad h \circ g = g \circ h \text{ for all } g \in S^1\}, \\ \infty, & \text{if } \{\ldots\} = \emptyset, \end{cases}$$

and let $\tilde{\tau}(\emptyset) = 0$.

The proof of Proposition 5.17 may be carried over easily to see that $\tilde{\tau}$ is an index for $(S^1, \mathcal{A}, \Gamma)$.

Multiple Periodic Orbits of Hamiltonian Systems

As an application, we present the following theorem on the existence of "many" periodic solutions of Hamiltonian systems, due to Ekeland and Lasry [1]:

5.18 Theorem. *Suppose $H \in C^1(\mathbb{R}^{2n}; \mathbb{R})$, and for some $\beta > 0$ assume that $C = \{x \in \mathbb{R}^{2n} \ ; \ H(x) \leq \beta\}$ is strictly convex, with boundary $S = \{x \in \mathbb{R}^{2n} \ ; \ H(x) = \beta\}$ satisfying $x \cdot \nabla H(x) > 0$ for $x \in S$. Suppose that for numbers $r, R > 0$ with*

$$r < R < \sqrt{2}\, r$$

we have

$$B_r(0) \subset C \subset B_R(0) \ .$$

Then there exist at least n distinct periodic solutions of the equation

$$\dot{x} = \mathcal{J} \nabla H(x) \tag{5.1}$$

on S. ($\mathcal{J}$ is defined on p. 60.)

Theorem 5.18 provides a "global" analogue of a result by Weinstein [1] on the existence of periodic orbits of Hamiltonian systems near an equilibrium. Further extensions and generalizations of Theorem 5.18 were given by Ambrosetti-Mancini [2] and Berestycki-Lasry-Mancini-Ruf [1] who allow for energy surfaces "pinched" between ellipsoids rather than spheres. Moreover, Ekeland-Lassoued [1] have been able to show that any strictly convex energy surface carries at least two distinct periodic orbits of (5.1). It is conjectured that a result like Theorem 5.18 holds true in general on such surfaces; the proof of this conjecture, however, remains open.

Proof of Theorem 5.18. We follow Ambrosetti and Mancini [2]. As observed in Section I.6 we may assume that

$$H(sx) = s^q H(x) \tag{5.2}$$

is homogeneous of degree q, $1 < q < 2$, and strictly convex. Moreover, dividing H by βq we may assume that $\beta = \frac{1}{q}$. By our assumption on S and (5.2), finally, we have

$$\frac{1}{qR^q}|x|^q \le H(x) \le \frac{1}{qr^q}|x|^q \ . \tag{5.3}$$

Let H^* be the Legendre-Fenchel transform of H as in I.6. Recall that $H^* \in C^1$, and $H^*(sy) = s^p H^*(y)$ with $p = \frac{q}{q-1} > 2$; moreover, (5.3) translates into

$$\frac{1}{p}r^p|y|^p \le H^*(y) \le \frac{1}{p}R^p|y|^p \ . \tag{5.4}$$

Also let K be the integral operator

$$(Ky)(t) = \int_0^t \mathcal{J}y \, dt$$

on

$$L_0^p = \big\{ y \in L^p_{loc}\big(\mathbb{R}; \mathbb{R}^{2n}\big) \ ; \ y(t+2\pi) = y(t), \int_0^{2\pi} y \, dt = 0 \big\} \ .$$

Then – as described in detail in Section I.6, following Equations (I.6.5′), (I.6.6′) – a function $y \in L_0^p \setminus \{0\}$ solves the Equation (I.6.7), that is, the equation

$$\nabla H^*(y) - Ky = x_0 \tag{5.5}$$

for some $x_0 \in \mathbb{R}^{2n}$, if and only if the function $x = Ky + x_0 \in C^1([0, 2\pi]; \mathbb{R}^{2n})$ solves (5.1) with $H\big(x(t)\big) =: h/q > 0$, which in turn implies that $\tilde{x}(t) = h^{-1/q} x\left(h^{2/q-1}t\right)$ solves (5.1) with $H\big(\tilde{x}(t)\big) \equiv \beta = \frac{1}{q}$.

Suppose we can exhibit n distinct solutions y_k of (5.5) with minimal period 2π corresponding to distinct solutions x_k of (5.1) with energies $H\big(x_k(t)\big) = h_k$. Then either $h_j = h_k$ and the corresponding $\tilde{x}_j \ne \tilde{x}_k$ (since the solutions x_k are distinct). Or $h_j \ne h_k$ and $\tilde{x}_j$, $\tilde{x}_k$ will have different minimal period and hence be distinct. In any event we will have achieved the proof of the theorem. This is the strategy that we now follow.

Denote $E\colon L_0^p \to \mathbb{R}$ the dual variational integral corresponding to (5.5), given by

$$E(y) = \int_0^{2\pi} \left(H^*(y) - \frac{1}{2}\langle y, Ky\rangle \right) dt \ .$$

Note that we have an S^1-action on L_0^p, via

$$(\tau, y) \mapsto y_\tau(t) = y(t+\tau) \ , \qquad \text{for all } \gamma = e^{i\tau} \in S^1 \ .$$

This action leaves E invariant. Moreover, y has minimal period 2π if and only if $y_\tau = y$ precisely for $\tau \in 2\pi\mathbb{Z}$. Denote

$$m = \inf \big\{ E(y) \ ; \ y \in L_0^p \big\}$$

and let

$$m^* = \inf\left\{E(y)\ ;\ y \in L_0^p,\ \exists k \in \mathbb{N}, k \geq 2 :\ y_\tau = y \text{ for } \tau = 2\pi/k\right\}\ .$$

Observe that, since $p > 2$ and since the spectrum of K contains positive eigenvalues, we have $m \leq m^* < 0$. Also note that the S^1-action will be free on the set

$$M = \{y \in L_0^p\ ;\ m \leq E(y) < m^*\}\ .$$

In particular, any $y \in M$ will have minimal period 2π.

Finally, E satisfies condition (P.-S.). Indeed, since E is coercive on L_0^p, any (P.-S.)-sequence (y_m) for E is bounded in L_0^p. Thus we may assume that $y_m \rightharpoonup y$ weakly in L_0^p and $Ky_m \to Ky$ strongly in $L^q\big([0,2\pi];\mathbb{R}^{2n}\big)$. Recall from Section I.6 that by strict convexity and homogeneity of H^* the differential ∇H^* is strongly monotone. That is, we have

$$\begin{aligned} o(1)\|y_m - y\|_{L^p} &= \langle y_m - y, DE(y_m) - DE(y)\rangle \\ &= \int_0^{2\pi} \langle y_m - y, \nabla H^*(y_m) - \nabla H^*(y)\rangle - \langle y_m - y, Ky_m - Ky\rangle\ dt \\ &\geq \alpha\big(\|y_m - y\|_{L^p}\big)\|y_m - y\|_{L^p} - o(1)\ , \end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$, with a non-negative, non-decreasing function α that vanishes only at 0. If follows that $y_m \to y$ strongly in L_0^p, as claimed.

Thus, by Theorem 5.11, the proof will be complete if we can show that the (simplified) S^1-index of a suitable compact S^1-invariant subset of M is $\geq n$. (From the definition of M it is clear that any pseudo-gradient flow on M will be complete in forward time.)

Note that since E is weakly lower semi-continuous and coercive on L_0^p and since in both cases the set of comparison functions is weakly closed, m and m^* will be attained in their corresponding classes. Let $y^* \in L_0^p$ satisfy

$$E(y^*) = m^*,\ y_\tau^* = y^* \text{ for some } \tau \notin 2\pi\mathbb{Z}\ .$$

By minimality of y^*

$$\langle y^*, DE(y^*)\rangle = p\int_0^{2\pi} H^*(y^*)\ dt - \int_0^{2\pi} \langle y^*, Ky^*\rangle\ dt = 0\ ,$$

whence in particular

$$m^* = \left(\frac{1}{p} - \frac{1}{2}\right)\int_0^{2\pi} \langle y^*, Ky^*\rangle\ dt\ .$$

We may assume that $\tau = \frac{2\pi}{k}$ for some $k \in \mathbb{N},\ k > 1$. Hence we obtain as comparison function

$$y(t) = y^*\left(\frac{t}{k}\right) \in L_0^p\ ,$$

and

$$m \le \inf_{s>0} E(sy) = \inf_{s>0} \left(s^p \int_0^{2\pi} H^*(y)\, dt - \frac{s^2}{2} \int_0^{2\pi} \langle y, Ky \rangle\, dt \right) .$$

But

$$\int_0^{2\pi} H^*(y)\, dt = \int_0^{2\pi} H^*(y^*)\, dt = \frac{1}{p} \int_0^{2\pi} \langle y^*, Ky^* \rangle\, dt ,$$

while

$$\int_0^{2\pi} \langle y, Ky \rangle\, dt = k \int_0^{2\pi} \langle y^*, Ky^* \rangle\, dt . \tag{5.6}$$

Hence

$$\begin{aligned} m &\le \inf_{s>0} \left(\frac{s^p}{p} - \frac{s^2}{2} k \right) \int_0^{2\pi} \langle y^*, Ky^* \rangle\, dt \\ &= k^{\frac{p}{p-2}} \left(\frac{1}{p} - \frac{1}{2} \right) \int_0^{2\pi} \langle y^*, Ky^* \rangle\, dt \\ &\le 2^{\frac{p}{p-2}} m^* < 0 . \end{aligned} \tag{5.7}$$

To obtain a lower bound on m, let $y \in L_0^p$ satisfy

$$E(y) = m .$$

Then

$$\langle y, DE(y) \rangle = p \int_0^{2\pi} H^*(y)\, dt - \int_0^{2\pi} \langle y, Ky \rangle\, dt = 0 , \tag{5.8}$$

and hence

$$m = \left(\frac{1}{p} - \frac{1}{2} \right) \int_0^{2\pi} \langle y, Ky \rangle\, dt = \left(1 - \frac{p}{2} \right) \int_0^{2\pi} H^*(y)\, dt . \tag{5.9}$$

Note that by (5.4) for $z \in L_0^p$ we have

$$\frac{1}{2\pi} \int_0^{2\pi} H^*(z)\, dt \ge \frac{r^p}{2\pi p} \int_0^{2\pi} |z|^p\, dx \ge \frac{r^p}{p} \left(\frac{1}{2\pi} \int_0^{2\pi} |z|^2\, dx \right)^{p/2} . \tag{5.10}$$

Let $\Sigma = \{ z \in L_0^p \ ; \ \frac{1}{2\pi} \int_0^{2\pi} |z|^2\, dt = 1 \}$ and denote

$$b = \sup \left\{ \int_0^{2\pi} \langle z, Kz \rangle\, dt \ ; \ z \in \Sigma \right\} > 0 .$$

Then, if we let $y = \lambda z, \ z \in \Sigma$, by (5.8), (5.10) we have

$$\begin{aligned} \lambda^{2-p} b &\ge \lambda^{2-p} \int_0^{2\pi} \langle z, Kz \rangle\, dt = p \int_0^{2\pi} H^*(z)\, dt \\ &\ge 2\pi r^p . \end{aligned} \tag{5.11}$$

Since $p > 2$ this implies a bound from above for λ. By estimating

$$\begin{aligned} m &= \left(\frac{1}{p} - \frac{1}{2}\right) \lambda^2 \int_0^{2\pi} \langle z, Kz\rangle \, dt \geq \left(\frac{1}{p} - \frac{1}{2}\right) \lambda^2 b \\ &\geq \left(\frac{1}{p} - \frac{1}{2}\right) (2\pi)^{-2/(p-2)} b^{p/(p-2)} r^{-2p/(p-2)} =: c_0 r^{-2p/(p-2)} \end{aligned} \tag{5.12}$$

with a constant $c_0 < 0$, the latter translates into a lower bound for m, and hence for m^*, by (5.7). Now let

$$\Sigma_n = \Big\{ z(t) = e^{\mathcal{J}t}(\xi, \eta) = (\xi \cos t - \eta \sin t, \, \xi \sin t + \eta \cos t) \; ; \\ (\xi, \eta) \in \mathbb{R}^{2n} \text{ with } |\xi|^2 + |\eta|^2 = 1 \Big\} \subset \Sigma \; .$$

Clearly Σ_n is S^1-invariant. Moreover, note that for $z \in \Sigma_n$ we have $H^*\big(z(t)\big) \leq R^p/p$ for all t, whence

$$\frac{1}{2\pi} \int_0^{2\pi} H^*(z) \, dt \leq R^p/p \; . \tag{5.13}$$

Now, we have

Lemma 5.19. *$\int_0^{2\pi} \langle z, Kz\rangle \, dt = b$ for any $z \in \Sigma_n$.*

Postponing the proof of Lemma 5.19, we conclude the proof of Theorem 5.18 as follows. For $z \in \Sigma_n$ let $\lambda = \lambda(z) > 0$ satisfy

$$p\lambda^p \int_0^{2\pi} H^*(z) \, dt - \lambda^2 \int_0^{2\pi} \langle z, Kz\rangle \, dt = 0 \; . \tag{5.14}$$

By (5.10) the map $z \mapsto \lambda(z) z$ is an S^1-equivariant C^1-embedding of Σ_n into L_0^p, mapping Σ_n onto an S^1-invariant set $\tilde{\Sigma}_n$ diffeomorphic to Σ_n by radial projection; in particular, the (simplified) S^1-index $\tilde{\tau}(\tilde{\Sigma}_n) \geq \tilde{\tau}(\Sigma_n)$.

From (5.13), (5.14) and Lemma 5.19, as in (5.11) we now obtain that for $z \in \Sigma_n$, $\lambda = \lambda(z)$ there holds

$$\lambda^{2-p} b = \lambda^{2-p} \int_0^{2\pi} \langle z, Kz\rangle \, dt = p \int_0^{2\pi} H^*(z) \, dt \leq 2\pi R^p \; .$$

Hence analogous to (5.12) we obtain

$$\sup_{y \in \tilde{\Sigma}_n} E(y) \leq c_0 R^{-2p/p-2} < 0 \; .$$

Since by assumption $R < \sqrt{2}\, r$, and in view of (5.7), this implies that

$$\sup_{y \in \tilde{\Sigma}_n} E(y) < \frac{1}{2^{p/(p-2)}} c_0 r^{-2p/(p-2)} \leq \frac{1}{2^{p/p-2}} m \leq m^* \; ;$$

that is, $\tilde{\Sigma}_n \subset M$. Hence the proof of Theorem 5.18 is complete if we show that $\tilde{\tau}(\Sigma_n) \geq n$.

But any S^1-equivariant map $h\colon \Sigma_n \to \mathbb{C}^m \setminus \{0\}$ with $h \circ g = g \circ h$ induces an odd map of $S^{2n-1} = \{(\xi,\eta) \in \mathbb{R}^{2n} \,;\, |\xi|^2 + |\eta|^2 = 1\}$ into $\mathbb{C}^m \setminus \{0\} \cong \mathbb{R}^{2m} \setminus \{0\}$, given by

$$(\xi,\eta) \mapsto h\big(e^{\mathcal{J}t}(\xi,\eta)\big) \ .$$

By Proposition 5.2 we conclude that $2m \geq 2n$, whence $\tilde{\tau}(\Sigma_n) \geq n$. (Since the map

$$z = z(t) = e^{\mathcal{J}t}(\xi,\eta) \mapsto z(0) = (\xi,\eta) \in \mathbb{R}^{2n} \setminus \{0\} \cong \mathbb{C}^n \setminus \{0\}$$

is S^1-equivariant, we actually have equality $\tilde{\tau}(\Sigma_n) = n$.) □

Proof of Lemma 5.19. Since K is compact there exists $z \in \Sigma$ satisfying

$$\int_0^{2\pi} \langle z, Kz \rangle \, dt = b \ .$$

By (5.6), z must have minimal period 2π.

Moreover, z satisfies

$$Kz + x_0 = \lambda z$$

for some $\lambda \in \mathbb{R}$, where $x_0 = -\frac{1}{2\pi} \int_0^{2\pi} K(z) \, dt$. Setting

$$x = Kz + x_0,$$

equivalently, x solves

$$\lambda \dot{x} = \mathcal{J}x \ .$$

Hence, $|x| \equiv const.$ Moreover, the properties of z imply that $x \neq 0$ and that x has minimal period 2π. A scalar multiple of x (and hence also a scalar multiple of z) thus belong to Σ_n.

By S^1-invariance of $I(z) = \int_0^{2\pi} \langle z, Kz \rangle \, dt$ and invariance of I under rotations in $\mathbb{R}^{2n}$, if $I(z) = b$ for *some* $z \in \Sigma_n$, it follows that $I(z) = b$ for *all* $z \in \Sigma_n$. The proof is complete. □

6. The Mountain Pass Lemma and its Variants

The minimax principle and its variants essentially cover all possibilities how existence results for saddle points can be drawn from information about the topology of the sub-level sets of a functional E.

However, unless the domain of E itself has a rich topology, finding the right notion of flow-invariant family may be quite tiresome. Fortunately, there are existence results for saddle points tailor-made for applications. These are the famous (infinite dimensional) mountain pass lemma and its variants, due to Ambrosetti and Rabinowitz [1]. The simplest form of these results reads as follows.

6.1 Theorem. *Suppose $E \in C^1(V)$ satisfies (P.-S.). Assume that*
(1°) $E(0) = 0$;
(2°) $\exists \rho > 0,\ \alpha > 0 :\ \|u\| = \rho \Rightarrow E(u) \geq \alpha$;
(3°) $\exists u_1 \in V :\ \|u_1\| \geq \rho$ and $E(u_1) < \alpha$.
Define

$$P = \left\{ p \in C^0\left([0,1]; V\right)\ ;\ p(0) = 0,\ p(1) = u_1 \right\} .$$

Then

$$\beta = \inf_{p \in P}\ \sup_{u \in p} E(u) \geq \alpha$$

is a critical value.

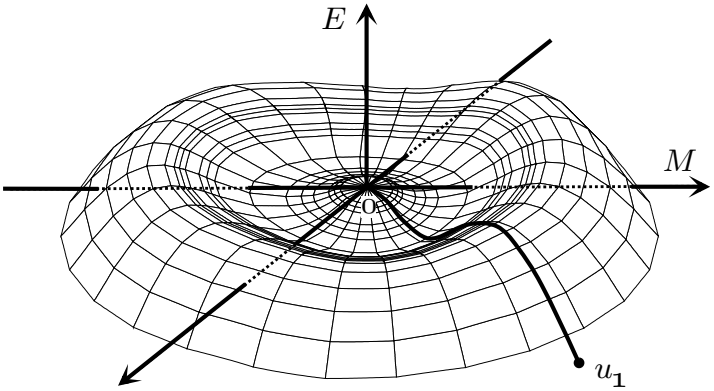

Fig. 6.1. On the mountain pass lemma of Ambrosetti and Rabinowitz

Proof. Suppose by contradiction that $K_\beta = \emptyset$. For $\overline{\varepsilon} = \min\{\alpha, \alpha - E(u_1)\}$ and $N = \emptyset$ determine $\varepsilon > 0$ and a deformation Φ as in Theorem 3.4. By definition of β, there exists $p \in P$ such that

$$\sup_{u \in p} E(u) < \beta + \varepsilon\ .$$

Consider $p_1 = \Phi(p, 1)$. Note that by choice of $\overline{\varepsilon}$ the deformation $\Phi(\cdot, 1)$ leaves $u_0 = 0$ and u_1 fixed. Hence $p_1 \in P$. Moreover, $\Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon}$; therefore

$$\sup_{u \in p_1} E(u) = \sup_{u \in p} E\left(\Phi(u, 1)\right) < \beta - \varepsilon\ ,$$

by choice of p. But this contradicts the definition of β, and the proof is complete. □

Remark. Observe that by Remark 3.5(3°) the proof, and hence the assertion of Theorem 6.1, remain true if we only require (P.-S.) at the level β.

Applications to Semilinear Elliptic Boundary Value Problems

Theorem 6.1 permits an alternative proof of Theorem I.2.1. However, Theorem 6.1 can also be applied to more general problems of the type

$$-\Delta u = g(\cdot, u) \qquad \text{in } \Omega \,, \tag{6.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \,, \tag{6.2}$$

which cannot be solved by a constrained minimization method.

6.2 Theorem. *Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, $n \geq 3$, and let $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with primitive $G(x,u) = \int_0^u g(x,v)\, dv$. Suppose that the following conditions hold:*
(1°) $g(x,0) = 0$ and $\limsup_{u\to 0} \frac{g(x,u)}{u} \leq 0$, uniformly in $x \in \Omega$;
(2°) $\exists p < 2^* = \frac{2n}{n-2}$, $C\colon |g(x,u)| \leq C\big(1+|u|^{p-1}\big)$, for almost every $x \in \Omega$, $u \in \mathbb{R}$;
(3°) $\exists q > 2$, $R_0 : 0 < q\,G(x,u) \leq g(x,u)\,u$, for almost every $x \in \Omega$, if $|u| \geq R_0$.
Then problem (6.1), (6.2) admits non-trivial solutions $u^+ \geq 0 \geq u_-$. If, in addition, g is Hölder continuous in both variables, then $u^+ > 0 > u_-$ in Ω.

Remark. Here again, $G(x,u) = \int_0^u g(x,v)\, dv$ denotes a primitive of g. An analogous result is valid for $n = 2$. However, notation is simpler if we consider only $n \geq 3$. Wang, Z.Q. [1], under (essentially) the assumptions of Theorem 6.2, recently has established the existence of a third non-trivial solution. (Note that by (1°) problem (6.1), (6.2) always admits the solution $u = 0$.)

Proof. Problem (6.1), (6.2) corresponds to the Euler-Lagrange equation of the functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega G(x,u)\, dx$$

on the space $H_0^{1,2}(\Omega)$. Assumption (2°) implies that E is of class C^1.

To see that E satisfies (P.-S.) , first note that by (2°) the map $u \mapsto g(\cdot,u)$ takes bounded sets in $L^p(\Omega)$ into bounded sets in $L^{p/(p-1)}(\Omega) \subset H^{-1}(\Omega)$. Therefore, and since for $p < \frac{2n}{n-2}$ by Rellich's theorem the space $H_0^{1,2}(\Omega)$ embeds into $L^p(\Omega)$ compactly, the map $K\colon H_0^{1,2}(\Omega) \to H^{-1}(\Omega)$, given by $K(u) = g(\cdot,u)$, is compact. Since DE is of the form $DE(u) = -\Delta u - g(\cdot,u)$, by Proposition 2.2 it suffices to show that any (P.-S.)-sequence (u_m) for E is bounded in $H_0^{1,2}(\Omega)$.

Let (u_m) be a (P.-S.)-sequence. Then we obtain

$$\begin{aligned} C + o(1)\|u_m\|_{H_0^{1,2}} &\geq q\,E(u_m) - \langle u_m, DE(u_m)\rangle \\ &= \frac{q-2}{2}\int_\Omega |\nabla u_m|^2\, dx + \int_\Omega \big(g(x,u_m)u_m - qG(x,u_m)\big)\, dx \\ &\geq \frac{q-2}{2}\|u_m\|^2_{H_0^{1,2}} + \mathcal{L}^n(\Omega)\cdot \operatorname*{ess\,inf}_{x\in\Omega, v\in\mathbb{R}} \big(g(x,v)v - qG(x,v)\big) \,, \end{aligned} \tag{6.3}$$

where $o(1) \to 0$ $(m \to \infty)$.

But by (2°) and (3°) the last term is finite and the desired conclusion follows.

Next observe that $E(0) = 0$; moreover, by (1°), (2°) for any $\varepsilon > 0$ there exists $C(\varepsilon)$ such that

$$G(x,u) \le \varepsilon |u|^2 + C(\varepsilon)|u|^p$$

for all $u \in \mathbb{R}$ and almost every $x \in \Omega$. It follows that

$$\begin{aligned} E(u) &\ge \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \varepsilon \int_\Omega |u|^2\,dx - C(\varepsilon)\int_\Omega |u|^p\,dx \\ &\ge \left(\frac{1}{2} - \frac{\varepsilon}{\lambda_1} - C(\varepsilon)\|u\|^{p-2}_{H^{1,2}_0}\right)\|u\|^2_{H^{1,2}_0} \ge \alpha > 0\ , \end{aligned}$$

provided $\|u\|_{H^{1,2}_0} = \rho$ is sufficiently small. Here λ_1 denotes the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions, given by the Rayleigh-Ritz quotient (I.2.4). Moreover, we have used the Sobolev embedding $H^{1,2}_0(\Omega) \hookrightarrow L^p(\Omega)$; see Theorem A.5 of Appendix A.

Finally, condition (3°) can be restated as a differential inequality for the function G, of the form

$$u|u|^q \frac{d}{du}\big(|u|^{-q}G(x,u)\big) \ge 0\ ,\ \text{for } |u| \ge R_0\ .$$

Upon integration we infer that for $|u| \ge R_0$ we have

$$G(x,u) \ge \gamma_0(x)|u|^q \tag{6.4}$$

with $\gamma_0(x) = R_0^{-q}\min\big\{G(x,R_0),\ G(x,-R_0)\big\} > 0$. Hence, if $u \in H^{1,2}_0(\Omega)$ does not vanish identically, we obtain that with constants $C(u)$, $c(u) > 0$ there holds

$$\begin{aligned} E(\lambda u) &= \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(x,\lambda u)\,dx \\ &\le C(u)\,\lambda^2 - c(u)\lambda^q + \mathcal{L}^n(\Omega)\operatorname*{ess\,sup}_{x\in\Omega,|v|\le R_0}|G(x,v)|\ , \\ &\to -\infty \qquad \text{as } \lambda \to \infty\ . \end{aligned} \tag{6.5}$$

But then, for fixed $u \ne 0$ and sufficiently large $\lambda > 0$ we may let $u_1 = \lambda u$, and from Theorem 6.1 we obtain the existence of a non-trivial solution u to (6.1), (6.2).

In order to obtain solutions $u^+ \ge 0 \ge u_-$, we may truncate g above or below $u = 0$, replacing g by 0, if $u \le 0$, to obtain a positive solution, respectively replacing g by 0, if $u \ge 0$, in order to obtain a solution $u_- \le 0$. Denote the truncated functions by $g^\pm(x,u)$, with primitive $G^\pm(x,u) = \int_0^u g^\pm(x,v)\,dv$. Note that (1°), (2°) remain valid for $g^\pm$ while (3°) will hold for $\pm u \ge R_0$, almost everywhere in Ω. Moreover, for $\pm u \le 0$ all terms in (3°) vanish.

Denote

$$E^\pm(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G^\pm(x,u)\,dx$$

the functional related to $g^{\pm}$. Then as above $E^{\pm} \in C^1(H_0^{1,2}(\Omega))$ satisfies (P.-S.), $E^{\pm}(0) = 0$, and condition (2°) of Theorem 6.1 holds. Moreover, choosing a comparison function $u > 0$, and letting $u_1 = \lambda u$ for large positive, respectively negative λ, by (6.4), (6.5), also condition (3°) of Theorem 6.1 is satisfied. Our former reasoning then yields a non-trivial critical point $u^{\pm}$ of $E^{\pm}$, weakly solving the equation

$$-\Delta u^{\pm} = g^{\pm}(\cdot, u^{\pm}) \qquad \text{in } \Omega \ .$$

Since $g^{\pm}(u) = 0$ for $\pm u \leq 0$, by the weak maximum principle $\pm u^{\pm} \geq 0$; see Theorem B.5 of Appendix B. Hence the functions $u^{\pm}$ in fact are weak solutions of the original Equation (6.1).

Finally, if g is Hölder continuous, by (2°) and Lemma B.3 of Appendix B and the remarks following it, $u^{\pm} \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Hence the strong maximum principle gives $u^+ > 0 > u_-$, as desired. (For a related problem a similar reasoning was used by Ambrosetti-Lupo [1].) □

The Symmetric Mountain Pass Lemma

More generally, for problems which are invariant under the involution $u \mapsto -u$, we expect the existence of infinitely many solutions, as in the case of problem (I.2.1), (I.2.3); see Theorem 5.8. However, if a problem does not exhibit the particular homogeneity of problem (I.2.1), (I.2.3), in general it cannot be reduced to a variational problem on the unit sphere in L^p, and a global method is needed. Fortunately, there is a "higher-dimensional" version of Theorem 6.1, especially adapted to functionals with a $\mathbb{Z}_2$-symmetry, the symmetric mountain pass lemma – again due to Ambrosetti-Rabinowitz [1]:

6.3 Theorem. *Suppose $E \in C^1(V)$ is even, that is $E(u) = E(-u)$, and satisfies (P.-S.). Let $V^+, V^- \subset V$ be closed subspaces of V with* codim $V^+ \leq$ dim $V^- < \infty$ *and suppose $V = V^- + V^+$. Also suppose there holds*
(1°) $E(0) = 0$,
(2°) $\exists \alpha > 0,\ \rho > 0\ \forall u \in V^+ : \|u\| = \rho \ \Rightarrow\ E(u) \geq \alpha$,
(3°) $\exists R > 0\ \forall u \in V^- : \|u\| \geq R \ \Rightarrow\ E(u) \leq 0$.
Then for each $j,\ 1 \leq j \leq k =$ dim V^- − codim V^+ *the numbers*

$$\beta_j = \inf_{h \in \Gamma} \sup_{u \in V_j} E\,(h(u))$$

are critical, where

$$\Gamma = \{h \in C^0(V; V); h \text{ is odd}, \quad h(u) = u \text{ if } u \in V^- \text{ and } \|u\| \geq R\} \ ,$$

and where $V_1 \subset V_2 \subset \ldots V_k = V^-$ are fixed subspaces of dimension

$$\text{dim } V_j = \text{codim } V^+ + j \ .$$

Moreover, $\beta_k \geq \beta_{k-1} \geq \ldots \geq \beta_1 \geq \alpha$.

For the proof of Theorem 6.3 we need a topological lemma.

6.4 Intersection Lemma. *Let V, V^+, V^-, Γ, V_j, R be as in Theorem 6.3. Then for any $\rho > 0$, and any $h \in \Gamma$ there holds*

$$\gamma\big(h(V_j) \cap S_\rho \cap V^+\big) = j \ ,$$

where S_ρ denotes the ρ-sphere $S_\rho = \{u \in V; \|u\| = \rho\}$, and γ denotes the Krasnoselskii genus introduced in Section 5.1; in particular $h(V_j) \cap S_\rho \cap V^+ \neq \emptyset$.

Proof. Denote $S_\rho^+ = S_\rho \cap V^+$. For any $h \in \Gamma$ the set $A = h(V_j) \cap S_\rho^+$ is symmetric, compact, and $0 \notin A$. Thus by Proposition 5.4(5°) there exists a neighborhood U of A such that $\gamma(\overline{U}) = \gamma(A)$. Then

$$\begin{aligned} \gamma\left(h(V_j) \cap S_\rho^+\right) &\geq \gamma\left(h(V_j) \cap S_\rho \cap \overline{U}\right) \\ &\geq \gamma\left(h(V_j) \cap S_\rho\right) - \gamma\left(h(V_j) \cap S_\rho \setminus U\right) \ . \end{aligned}$$

Let $Z \subset V^+$ be a direct complement of V^- and denote by $\pi: V \to V^-$ the continuous linear projection onto V^- associated with the decomposition $V = V^- \oplus Z$. Then, since U is a neighborhood of $h(V_j) \cap S_\rho^+$, we have $\pi\left(h(V_j) \cap S_\rho \setminus U\right) \not\ni 0$, and hence

$$\gamma\left(h(V_j) \cap S_\rho \setminus U\right) \leq \dim V^- < \infty \ .$$

On the other hand, by Proposition 5.4(2°), (4°)

$$\gamma\left(h(V_j) \cap S_\rho\right) \geq \gamma\left(h^{-1}(S_\rho) \cap V_j\right) \ .$$

But $h(0) = 0$, $h = id$ on $V_j \setminus B_R(0)$; hence $h^{-1}(S_\rho) \cap V_j$ bounds a symmetric neighborhood of the origin in V_j. Thus, by Proposition 5.2 we conclude that

$$\gamma\left(h(V_j) \cap S_\rho\right) \geq \dim V_j = \dim V^- + j \ ,$$

which implies the lemma. □

Proof of Theorem 6.3. By Lemma 6.4 we have $\beta_j \geq \alpha$ for all $j \in \{1, \ldots, k\}$. Moreover, clearly $\beta_{j+1} \geq \beta_j$ for all j. Suppose by contradiction that β_j is regular for some $j \in \{1, \ldots, k\}$. For $\beta = \beta_j$, $N = \emptyset$, and $\overline{\varepsilon} = \alpha$ let $\varepsilon > 0$ and Φ be as constructed in Theorem 3.4. By Remark 3.5(4°) we may assume that $\Phi(\cdot, t)$ is odd for any $t \geq 0$.

Note that by choice of $\overline{\varepsilon}$ we have $\Phi(\cdot, t) \circ h \in \Gamma$ for any $h \in \Gamma$. Choose $h \in \Gamma$ such that there holds $E\left(h(u)\right) < \beta + \varepsilon$ for all $u \in V_j$. Then $h_1 = \Phi(\cdot, 1) \circ h \in \Gamma$, and for all $u \in V_j$ there holds

$$E\left(h_1(u)\right) = E\left(\Phi\left(h(u), 1\right)\right) < \beta - \varepsilon \ ,$$

contradicting the definition of β. □

Note that in contrast to Theorem 5.7, our derivation of Theorem 6.3 does not allow us to obtain optimal multiplicity results in the case of degenerate critical values $\beta_j = \beta_{j+1} = \ldots = \beta_k$. However, for most applications this defect does not matter. (Actually, by using a notion of pseudo-index, Bartolo-Benci-Fortunato [1; Theorem 2.4] have been able to prove optimal multiplicity results also in the case of degenerate critical values.) The following result is typical.

6.5 Theorem. *Suppose V is an infinite dimensional Banach space and suppose $E \in C^1(V)$ satisfies (P.-S.), $E(u) = E(-u)$ for all u, and $E(0) = 0$. Suppose $V = V^- \oplus V^+$, where V^- is finite dimensional, and assume the following conditions:*
(1°) $\exists \alpha > 0,\ \rho > 0 :\ \|u\| = \rho,\ u \in V^+ \Rightarrow E(u) \geq \alpha$.
(2°) For any finite dimensional subspace $W \subset V$ there is $R = R(W)$ such that $E(u) \leq 0$ for $u \in W,\ \|u\| \geq R$.
Then E possesses an unbounded sequence of critical values.

Proof. Choose a basis $\{\varphi_1, \ldots\}$ for V^+ and for $k \in \mathbb{N}$ let $W_k = V^- \oplus \text{span } \{\varphi_1, \ldots, \varphi_k\}$, with $R_k = R(W_k)$. Since $W_k \subset W_{k+1}$, we may assume that $R_k \leq R_{k+1}$ for all k.

Define classes

$$\begin{aligned}\Gamma_k = \big\{ & h \in C^0(V;V)\ ;\ h \text{ is odd,} \\ & \forall j \leq k,\ u \in W_j\ :\ \|u\| \geq R_j \Rightarrow h(u) = u \big\}\end{aligned}$$

and let

$$\beta_k = \inf_{h \in \Gamma_k} \sup_{u \in W_k} E\left(h(u)\right)\ .$$

Then, by the argument of Theorem 6.3, each of these numbers defines a critical value $\beta_k \geq \alpha$ of E. Indeed, for each k we let W_k take the role of V^- in Theorem 6.3 and consider the case $j = k$ with $V_k = W_k$. Observe that for the pseudo-gradient flow Φ constructed in the proof of Theorem 6.3 there holds $\Phi(\cdot, t) \in \Gamma_k$ for all t. Moreover,

$$\begin{aligned}\Gamma_k \subset \Gamma = \big\{ & h \in C^0(V;V); h \text{ is odd, } h(u) = u \\ & \text{if } u \in W_k \text{ and } \|u\| \geq R_k \big\}.\end{aligned}$$

Hence also Lemma 6.4 remains true with Γ_k instead of Γ, and β_k is critical, as claimed.

The proof therefore will be complete when we show the following:

Assertion. *The sequence (β_k) is unbounded.*

First observe that, since $\Gamma_k \supset \Gamma_{k+1}$ while $W_k \subset W_{k+1}$ for all k, the sequence (β_k) is non-decreasing. Suppose by contradiction that $\sup_k \beta_k = \beta < \infty$. Note that $K = K_\beta$ is compact and symmetric. Moreover, since $\beta \geq \alpha > 0$ we also have $0 \notin K$. By Proposition 5.4(5°) then $\gamma(K) < \infty$, and there exists a symmetric neighborhood N of K such that $\gamma(\overline{N}) = \gamma(K_\beta)$.

Choose $\overline{\varepsilon} = \alpha > 0$, and let $\varepsilon \in]0, \overline{\varepsilon}[$ and Φ be determined according to Theorem 3.4 corresponding to $\overline{\varepsilon}, \beta$, and N. Observe that Φ may be chosen to be odd, see Remark 3.5(4°), and that by choice of $\overline{\varepsilon}$ for any $j \in \mathbb{N}$ we have $\Phi(u,t) = u$ for all $t \geq 0$ and any $u \in W_j$ such that $\|u\| \geq R_j$; that is, $\Phi(\cdot, t) \in \Gamma_k$ for any k.

Let $\beta = \beta^{(0)},\ \varepsilon = \varepsilon^{(0)},\ \Phi = \Phi_{\beta^{(0)}}$. We iterate the above procedure. For each number $\beta' \in [\alpha, \beta - \varepsilon]$ also the set $K' = K_{\beta'}$ is compact, has finite genus $\gamma(K_{\beta'})$, and possesses a neighborhood $N' = N_{\beta'}$ with $\gamma(\overline{N}') = \gamma(K')$. Moreover, for each such β', with $\overline{\varepsilon} = \alpha$ and $N = N_{\beta'}$ we may let $\varepsilon_{\beta'} \in]0, \overline{\varepsilon}[,\ \Phi_{\beta'}$ be determined according to Theorem 3.4. Finitely many neighborhoods $]\beta^{(l)} - \varepsilon^{(l)}, \beta^{(l)} + \varepsilon^{(l)}[,\ l = 1, \ldots, L$, where $\beta^{(l)} \in [\alpha, \beta - \varepsilon],\ \varepsilon^{(l)} = \varepsilon_{\beta^{(l)}}$, cover $[\alpha, \beta - \varepsilon]$. Clearly, we may assume

$$\beta = \beta^{(0)} \geq \beta^{(1)} \geq \ldots \geq \beta^{(l)} \geq \beta^{(l+1)} \geq \ldots \geq \beta^{(L)} \geq \alpha$$

and also that

$$\beta^{(l-1)} > \beta^{(l)} + \varepsilon^{(l)} > \beta^{(l-1)} - \varepsilon^{(l-1)} > \beta^{(l)}$$

for all $l,\ 1 \leq l \leq L$.

For any $k \in \mathbb{N}$, with $\varepsilon = \varepsilon^{(0)} > 0$ as above, now choose $h \in \Gamma_k$ satisfying

$$\sup_{u \in W_k} E\left(h(u)\right) < \beta_k + \varepsilon \leq \beta + \varepsilon\ .$$

Composing h with $\Phi = \Phi_{\beta^{(0)}}$ yields a map $\Phi(\cdot, 1) \circ h =: h' \in \Gamma_k$ with

$$h'(W_k) \subset E_{\beta - \varepsilon} \cup N\ .$$

Consider the compositions

$$\begin{aligned} H^{(m,m)} &= id,\ H^{(m,m-1)} = \Phi_{\beta^{(m)}}(\cdot, 1)\ , \\ H^{(m,l)} &= \Phi_{\beta^{(m)}}(\cdot, 1) \circ \ldots \circ \Phi_{\beta^{(l+1)}}(\cdot, 1), \qquad 0 \leq l < m - 1 < L \end{aligned}$$

and let $h^{(0)} = \Phi_{\beta^{(0)}}(\cdot, 1) \circ h = h',\ h^{(l)} = \Phi_{\beta^{(l)}}(\cdot, 1) \circ h^{(l-1)} = H^{(l,0)} \circ h^{(0)}$, for $l = 1, \ldots, L$. By Theorem 3.4, each $H^{(m,l)}$ is the composition of homeomorphisms, hence a homeomorphism itself.

By induction, letting $N^{(0)} = N,\ N^{(l)} = N_{\beta^{(l)}},\ l = 1, \ldots, L$, we have

$$h^{(m)}(W_k) \subset E_{\beta^{(m)} - \varepsilon^{(m)}} \bigcup_{0 \leq l \leq m} H^{(m,l)}\left(N^{(l)}\right)$$

for all $m,\ 0 \leq m \leq L$. Moreover, $h^{(m)} \subset \Gamma_k$ for all $m,\ 0 \leq m \leq L$. In particular, $h^{(L)} \in \Gamma_k$ and

$$h^{(L)}(W_k) \subset E_\alpha \bigcup_{0 \leq l \leq L} H^{(L,l)}\left(N^{(l)}\right)\ .$$

Let $S_\rho^+ = \partial B_\rho(0; V^+)$. Then by assumption (1°) we have

$$h^{(L)}(W_k) \cap S_\rho^+ \subset \bigcup_{0 \leq l \leq L} \left(H^{(L,l)}\left(N^{(l)}\right) \cap S_\rho^+\right) \subset \bigcup_{0 \leq l \leq L} H^{(L,l)}\left(N^{(l)}\right)\ .$$

By monotonicity and sub-additivity of the genus γ, see Proposition 5.4(2°), (3°), and since each map $H^{(L,l)}$ is a homeomorphism, this implies that

$$
\begin{aligned}
\gamma\left(h^{(L)}(W_k)\cap S_\rho^+\right) &\le \sum_{0\le l\le L}\gamma\left(H^{(L,l)}\left(\overline{N^{(l)}}\right)\right)\\
&= \sum_{0\le l\le L}\gamma\left(\overline{(N^{(l)}}\right) = \sum_{0\le l\le L}\gamma\left(K_{\beta^{(l)}}\right) =: k_0 < \infty\ ,
\end{aligned}
\tag{6.6}
$$

with k_0 independent of k.

On the other hand, by the Intersection Lemma 6.4, for any $k\in\mathbb{N}$ we have

$$
\gamma\left(h(W_k)\cap S_\rho^+\right)\ge k\ . \tag{6.7}
$$

This holds for any $h\in\Gamma_k$, in particular, for $h=h^{(L)}$.

Since k is arbitrary, this contradicts (6.6); hence the proof of Theorem 6.5 is complete. □

Application to Semilinear Equations with Symmetry

As an application we state the following existence result for problems of the type (6.1), (6.2) involving odd nonlinearities g. Results of this kind are well known; see the references in Section 5. Note that the symmetry assumption allows removal of any conditions on g near $u=0$.

6.6 Theorem. *Let Ω be a smoothly bounded domain in $\mathbb{R}^n$, $n\ge 3$, and let $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function with primitive $G(\cdot,u)=\int_0^u g(\cdot,v)\,dv$. Suppose:*
(1°) g is odd: $g(x,-u)=-g(x,u)$,
and conditions (2°),(3°) of Theorem 6.2 are satisfied, that is
(2°) $\exists p<2^*=\frac{2n}{n-2}$, $C:|g(x,u)|\le C\left(1+|u|^{p-1}\right)$ almost everywhere,
(3°) $\exists q>2$, $R_0: 0<q\,G(x,u)\le g(x,u)u$ for almost every x, $|u|\ge R_0$.
Then problem (6.1), (6.2) admits an unbounded sequence (u_k) of solutions $u_k\in H_0^{1,2}(\Omega)$.

Proof. As in the proof of Theorem 6.2, define

$$
E(u)=\frac{1}{2}\int_\Omega |\nabla u|^2\,dx-\int_\Omega G(x,u)\,dx\ .
$$

Hypothesis (2°) implies that E is Fréchet-differentiable on $H_0^{1,2}(\Omega)$. A computation similar to (6.3) shows that the assertion of the theorem is equivalent to the assertion that E admits an unbounded sequence of critical values.

To prove the latter we invoke Theorem 6.5. Note that by the proof of Theorem 6.2 the functional E satisfies (P.-S.); see (6.3). Moreover, since g is odd, E is even: $E(u)=E(-u)$. Finally, $E(0)=0$.

Denote $0<\lambda_1<\lambda_2\le\lambda_3\le\dots$ the eigenvalues of $-\Delta$ on Ω with homogeneous Dirichlet data, as usual, and let φ_j be the corresponding eigenfunctions.

We claim that for k_0 sufficiently large there exist $\rho>0$, $\alpha>0$ such that for all $u\in V^+:=\operatorname{span}\{\varphi_k\ ;\ k\ge k_0\}$ with $\|u\|_{H_0^{1,2}}=\rho$ there holds

$E(u) \geq \alpha$. Indeed, by (2°), Sobolev's embedding $H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$, and Hölder's inequality, for $u \in V^+$ we have (with constants C_1, C_2 independent of u)

$$\begin{aligned} E(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - C\int_\Omega |u|^p\,dx - C \\ &\geq \frac{1}{2}\|u\|^2_{H_0^{1,2}} - C\,\|u\|^r_{L^2}\|u\|^{p-r}_{L^{2^*}} - C \\ &\geq \left(\frac{1}{2} - C_1\lambda_{k_0}^{-r/2}\|u\|^{p-2}_{H_0^{1,2}}\right)\|u\|^2_{H_0^{1,2}} - C_2 \end{aligned} \tag{6.8}$$

where $\frac{r}{2} + \frac{p-r}{2^*} = 1$. In particular, $r = n(1 - \frac{p}{2^*}) > 0$, and we may let $\rho = 2\sqrt{(C_2+1)}$ and choose $k_0 \in \mathbb{N}$ such that

$$C_1\lambda_{k_0}^{-r/2}\rho^{p-2} \leq \frac{1}{4}$$

to achieve

$$E(u) \geq 1 =: \alpha\ ,\ \text{for all } u \in V^+\ \text{ with } \|u\|_{H_0^{1,2}} = \rho\ .$$

Now fix V^+ as above and denote $V^- = \operatorname{span}\{\varphi_j\ ;\ j < k_0\}$ its orthogonal complement.

Finally, on any finite dimensional subspace $W \subset H_0^{1,2}$, by (3°) and (6.4), (6.5) there exist constants $C_i = C_i(W) > 0$ such that

$$\sup_{u \in \partial B_R(0;W)} E(u) \leq C_1R^2 - C_2R^q + C_3 \to -\infty \tag{6.9}$$

as $R \to \infty$.

Hence, Theorem 6.5 guarantees the existence of an unbounded sequence of critical values

$$\beta_k = \inf_{h \in \Gamma_k}\ \sup_{u \in W_k} E\left(h(u)\right),\quad k \geq k_0\ ,$$

where $W_k = \operatorname{span}\{\varphi_j\ ;\ j \leq k\}$ and Γ_k is defined as in the proof of Theorem 6.5. The proof is complete. □

6.7 Remark. We can estimate the rate at which the sequence (β_k) diverges. Indeed, by the Intersection Lemma 6.4 , letting $V_k = \operatorname{span}\{\varphi_j\ ;\ j \geq k\}$, $W_k = \operatorname{span}\{\varphi_1 \ldots, \varphi_k\}$ as in the proof above, we have

$$h(W_k) \cap S_\rho \cap V_k \neq \emptyset$$

for any $\rho > 0$, any $k \in \mathbb{N}$. Hence, by (6.8), with $r = n\left(1 - \frac{p}{2^*}\right) = p - \frac{n(p-2)}{2}$ we obtain

$$\begin{aligned}\beta_k &\geq \sup_{\rho>0} \inf_{u\in S_\rho\cap V_k} E(u)\\ &\geq \sup_{\rho>0} \left(\frac{1}{2} - C_1\lambda_k^{-r/2}\rho^{p-2}\right)\rho^2 - C_2\\ &\geq c\lambda_k{}^{\frac{r}{p-2}} = c\lambda_k{}^{\left(\frac{p}{p-2}-\frac{n}{2}\right)}\ ,\end{aligned}$$

for k large. Finally, by the asymptotic formula (see Weyl [1], or Edmunds-Moscatelli [1])

$$\lambda_k k^{-2/n} \to c > 0 \qquad (k\to\infty)\ ,$$

and it follows that with a constant $c>0$ we have

$$\beta_k \geq ck^{\frac{2p}{n(p-2)}-1} \tag{6.10}$$

for large k. Estimate (6.10) will prove useful in the next section.

7. Perturbation Theory

A natural question, which even today is not adequately settled, is whether the symmetry of the functional is important for results like Theorem 6.5 to hold.

A partial answer for problems of the kind studied in Theorem 6.6 was independently obtained by Bahri-Berestycki [1] and Struwe [1] in 1980.

In abstract form, the variational principle underlying these results later was phrased by Rabinowitz [10]. It will be convenient to write $\tilde{E}$ for the non-symmetric functional and to reserve the notation E for its even symmetrization.

7.1 Theorem. *Suppose $\tilde{E}\in C^1(V)$ satisfies (P.-S.). Let $W\subset V$ be a finite dimensional subspace of V, $w^*\in V\setminus W$, and let $W^* = W\oplus\operatorname{span}\{w^*\}$; also let*

$$W^*_+ = \{w + tw^*\ ;\ w\in W,\ t\geq 0\}$$

denote the "upper half-space" in W^*. Suppose
(1°) $\tilde{E}(0) = 0$,
(2°) $\exists R>0\ \forall u\in W:\ \|u\|\geq R\Rightarrow \tilde{E}(u)\leq 0$,
(3°) $\exists R^*\geq R\ \forall u\in W^*:\ \|u\|\geq R^*\Rightarrow \tilde{E}(u)\leq 0$,
and let

$$\begin{aligned}\Gamma = \{h\in C^0(V,V)\ ;\ &h \text{ is odd, } h(u) = u \text{ if } \max\{\tilde{E}(u),\tilde{E}(-u)\}\leq 0\ ,\\ &\text{in particular, if } u\in W, \|u\|\geq R, \text{ or if } u\in W^*, \|u\|\geq R^*\}\ .\end{aligned}$$

Then, if

$$\beta^* = \inf_{h\in\Gamma}\sup_{u\in W^*_+}\tilde{E}(h^*(u)) > \beta = \inf_{h\in\Gamma}\sup_{u\in W}\tilde{E}\,(h(u))\geq 0\ ,$$

the functional $\tilde{E}$ possesses a critical value $\geq\beta^*$.

Proof. For $\gamma\in]\beta,\beta^*[$ let

$$\Lambda = \{h \in \Gamma \,;\; \tilde{E}\,(h(u)) \leq \gamma \text{ for } u \in W\}\ .$$

By definition of β, clearly $\Lambda \neq \emptyset$. Hence

$$\gamma^* = \inf_{h\in\Lambda} \sup_{u\in W_+^*} \tilde{E}\,(h(u)) \geq \beta^*$$

is well-defined. We contend that γ^* is critical.

Assume by contradiction that γ^* is regular and choose $\varepsilon > 0$, Φ according to Theorem 3.4, corresponding to γ^*, $\overline{\varepsilon} = \gamma^* - \gamma > 0$, $N = \emptyset$. Also select $h \in \Lambda$ with

$$\sup_{u\in W_+^*} \tilde{E}\,(h(u)) < \gamma^* + \varepsilon\ .$$

Define an odd map $h' \colon W^* \to V$

$$h'(u) = \begin{cases} \Phi\,(h(u),1)\,, & \text{if } u \in W_+^* \\ -\Phi\,(h(-u),1)\,, & \text{if } -u \in W_+^*\ . \end{cases}$$

Note that by choice of $\overline{\varepsilon}$ and since $h \in \Lambda$ we have

$$\Phi\,(h(-u),1) = h(-u) = -h(u) = -\Phi\,(h(u),1)$$

for $u \in W$. Hence h' is well-defined, odd, and continuous. Moreover, since $0 \leq \beta < \gamma < \gamma^* - \varepsilon$, the map $\Phi(\cdot,1)$ fixes any point u that satisfies $\tilde{E}(u) \leq 0$ and $\tilde{E}(-u) \leq 0$. By definition of Λ so does h, and hence the composition $\Phi(\cdot,1) \circ h$. It follows that h' may be extended to a map $h' \in \Lambda$. But now the estimate

$$\sup_{u\in W_+^*} \tilde{E}\,(h'(u)) = \sup_{u\in W_+^*} \tilde{E}\,(\Phi\,(h(u),1)) < \gamma^* - \varepsilon$$

yields the desired contradiction. □

Theorem 7.1 suggests comparing a non-symmetric functional $\tilde{E}$ satisfying the remaining hypotheses (1°), (2°) of Theorem 6.5 with its symmetrization

$$E(u) = \frac{1}{2}\left(\tilde{E}(u) + \tilde{E}(-u)\right).$$

Then Theorem 6.5 applies to E and we obtain an unbounded sequence (β_k) of critical values of E. Playing the speed of divergence $\beta_k \to \infty$ off against the perturbation from symmetry $E - \tilde{E}$, from Theorem 7.1 we can glean the existence of infinitely many critical points for functionals perturbed from symmetry. Stating a general theorem of this kind, however, seems to involve so many technical conditions that any reader would ask himself if these conditions can ever be met in practice. Therefore we prefer to present such an application immediately.

Applications to Semilinear Elliptic Equations

We return to the setting of problem (6.1), (6.2) under (essentially) the assumptions (1°)–(3°) of Theorem 6.6 studied previously. However, in order to make estimate (6.4) uniform in $x \in \Omega$, we assume that g is continuous in all its variables. For convenience we restate also the remaining conditions.

7.2 Theorem. *Let Ω be a smoothly bounded domain in $\mathbb{R}^n, n \geq 3$. Suppose*
(1°) $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and odd with primitive $G(x,u) = \int_0^u g(x,v)\,dv$;
(2°) $\exists p < 2^* = \frac{2n}{n-2}$, $C : |g(x,u)| \leq C\left(1 + |u|^{p-1}\right)$ almost everywhere;
(3°) $\exists q > 2$, $R_0 : 0 < q\, G(x,u) \leq g(x,u)u$ for almost every x, $|u| \geq R_0$.
Moreover, suppose that

$$\frac{2p}{n(p-2)} - 1 > \frac{q}{q-1}\ .$$

Then for any $f \in L^2(\Omega)$ the problem

$$\begin{aligned} -\Delta u &= g(\cdot,u) + f && \text{in } \Omega\ , \\ u &= 0 && \text{on } \partial\Omega\ , \end{aligned}$$

has an unbounded sequence of solutions $u_k \in H_0^{1,2}(\Omega)$, $k \in \mathbb{N}$.

Proof. The assertion of the theorem is equivalent to the claim that the functional

$$\tilde{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(\cdot,u)\,dx - \int_\Omega fu\,dx$$

on $H_0^{1,2}(\Omega)$ possesses an unbounded sequence of critical values.

Clearly, $\tilde{E}(0) = 0$, and on any finite dimensional subspace $W \subset H_0^{1,2}(\Omega)$ by (3°) and (6.4), (6.5) the estimate

$$\sup_{u \in \partial B_\rho(0;W)} \tilde{E}(u) \leq C_1\rho^2 - C_2\rho^q + C_3\rho + C_4 \to -\infty \qquad (\rho \to \infty) \tag{7.1}$$

holds with constants $C_i = C_i(W) > 0$. (The term $\int_\Omega fu\,dx$ was estimated by using Hölder's inequality.) To see (P.-S.), as in (6.3) in the proof of Theorem 6.2, for a (P.-S.) sequence (u_m) consider the estimate

$$\begin{aligned} C + o(1)\,\|u_m\|_{H_0^{1,2}} &\geq q\tilde{E}(u_m) - \langle u_m, D\tilde{E}(u_m)\rangle \\ &= q\,E(u_m) - \langle u_m, DE(u_m)\rangle - (q-1)\int_\Omega fu\,dx \\ &\geq \frac{q-2}{2}\|u_m\|^2_{H_0^{1,2}} - C\|u_m\|_{H_0^{1,2}} - C\ , \end{aligned}$$

from which boundedness of (u_m) in $H_0^{1,2}(\Omega)$ follows at once. (Note that the symmetrized functional $E(u) = \frac{1}{2}\big(\tilde{E}(u) + \tilde{E}(-u)\big)$ equals the functional E considered in Theorem 6.6.) Using Proposition 2.2, the existence of a strongly

convergent subsequence of (u_m) now is established exactly as in the proof of Theorem 6.2.

Let $\lambda_j,\ \varphi_j,\ V^+ = \text{span}\{\varphi_k\ ;\ k \geq k_0\},\ W_k = \text{span}\{\varphi_j\ ;\ j \leq k\}$ as in the proof of Theorem 6.6. By (7.1) we may choose a non-decreasing sequence (R_k) such that $\tilde{E}(u) \leq 0$ for all $u \in W_k$ with $\|u\|_{H_0^{1,2}} \geq R_k$; hence also $E(u) \leq 0$ for all such u. With this sequence (R_k) let (Γ_k) be defined as in Theorem 6.5 and for $k \geq k_0$ let

$$\beta_k = \inf_{h \in \Gamma_k} \sup_{u \in W_k} E\,(h(u)) \geq c\, k^{\frac{2p}{n(p-2)}-1} - \beta_0 \tag{7.2}$$

be the sequence of critical values of the symmetrized functional E constructed in Theorem 6.6, the lower estimate following from (6.10). Suppose that $\tilde{E}$ does not have any critical values larger than some number $\overline{\beta} - 1$. Fix $\overline{k} \geq k_0$ such that $\beta_k \geq \overline{\beta}$ for $k \geq \overline{k}$. We will use Theorem 7.1 to derive a uniform bound

$$\beta_k \leq C, \qquad \text{for all } k \geq \overline{k}\ ,$$

which will yield the desired contradiction to (7.2).

For $k \geq \overline{k}$ let $W = W_k,\ w^* = \varphi_{k+1},\ W^* = W_{k+1}$, and let Γ be defined as in Theorem 7.1, with $R = R_k,\ R^* = R_{k+1}$. Note that $\Gamma \subset \Gamma_{k+1} \subset \Gamma_k$. Since by assumption there are no critical values $\beta > \overline{\beta} - 1$ for $\tilde{E}$, from Theorem 7.1 we conclude that

$$\begin{aligned} \tilde{\beta}^* := \tilde{\beta}_k^* &= \inf_{h^* \in \Gamma} \sup_{u \in W_+^*} \tilde{E}\,(h^*(u)) \\ &= \tilde{\beta} := \tilde{\beta}_k = \inf_{h \in \Gamma} \sup_{u \in W} \tilde{E}\,(h(u))\ . \end{aligned}$$

From oddness of any map $h \in \Gamma$ and the estimate

$$E(u) = \frac{1}{2}\left(\tilde{E}(u) + \tilde{E}(-u)\right) \leq \max\{\tilde{E}(u), \tilde{E}(-u)\}\ ,$$

valid for all u, we also deduce that

$$\tilde{\beta}_k \geq \inf_{h \in \Gamma} \sup_{u \in W} E\big(h(u)\big) \geq \beta_k \geq \overline{\beta}\ ,\ \text{for all } k \geq \overline{k}\ . \tag{7.3}$$

Given $\varepsilon > 0$ (say $\varepsilon = 1$) choose a map $h^* \subset \Gamma$ such that

$$\sup_{u \in W} \tilde{E}\big(h^*(u)\big) \leq \sup_{u \in W_+^*} \tilde{E}\big(h^*(u)\big) < \tilde{\beta}_k^* + \varepsilon = \tilde{\beta}_k + \varepsilon\ .$$

Let $\|\cdot\|$ denote the norm in $H_0^{1,2}(\Omega)$, respectively $H^{-1}(\Omega)$. Consider a pseudo-gradient vector field v for E such that

$$\|v\| \leq 2,\ \langle v(u), DE(u)\rangle \geq \min\big\{1, \|DE(u)\|\big\}\|DE(u)\|\ .$$

Since E is even, we may assume v to be odd. Note that

$$\begin{aligned}\langle v(u), D\tilde{E}(u)\rangle &= \langle v(u), DE(u)\rangle + \langle v(u), D\tilde{E}(u) - DE(u)\rangle \\ &\geq \min\{1, \|DE(u)\|\}\,\|DE(u)\| - 2\|D\tilde{E}(u) - DE(u)\| \\ &\geq \min\{1, \|D\tilde{E}(u)\|\}\,\|D\tilde{E}(u)\| - 4\,\|D\tilde{E}(u) - DE(u)\|\ .\end{aligned}$$

That is, v also is a pseudo-gradient vector field for $\tilde{E}$ off $\tilde{N}_\delta$ for any $\delta \geq 8$, where for $\delta > 0$ the set $\tilde{N}_\delta$ is given by

$$\tilde{N}_\delta = \big\{u \in H_0^{1,2}(\Omega)\ ;\ \|D\tilde{E}(u) - DE(u)\| > \delta^{-1}\min\big\{1, \|D\tilde{E}(u)\|\big\}\|D\tilde{E}(u)\|\big\}\ .$$

Note that – unless $f \equiv 0$, which is trivial – $\tilde{N}_\delta$ is a neighborhood of the set of critical points of E and $\tilde{E}$, for any $\delta > 1$. Denote $N_\delta = \tilde{N}_\delta \cup \big(-\tilde{N}_\delta\big)$ the symmetrized sets $\tilde{N}_\delta$. Let $0 \leq \eta \leq 1$ be an even, locally Lipschitz function satisfying $\eta(u) = 0$ in N_{10} and $\eta(u) = 1$ off $N = N_{20}$. Also let φ be Lipschitz, $0 \leq \varphi \leq 1$, $\varphi(s) = 0$ for $s \leq 0$, $\varphi(s) = 1$ for $s \geq 1$, and let

$$e(u) = -\varphi\big(\max\{\tilde{E}(u), \tilde{E}(-u)\}\big)\eta(u)v(u)$$

denote the truncated pseudo-gradient vector field on $H_0^{1,2}(\Omega)$.

Then, if Φ denotes the (odd) pseudo-gradient flow for E induced by e, Φ will also be a pseudo-gradient flow for $\tilde{E}$ and will strictly decrease $\tilde{E}$ off N with "speed"

$$\frac{d}{dt}\tilde{E}\big(\Phi(u,t)\big)|_{t=0} \leq -\frac{1}{2}\min\big\{1, \|D\tilde{E}(u)\|^2\big\}\ , \qquad \text{if } \tilde{E}(u) \geq 1\ .$$

Moreover, $\Phi(\cdot, t) \in \Gamma$ for all $t \geq 0$.

Composing h^* with $\Phi(\cdot, t)$, unless $\tilde{E}$ achieves its supremum on $\Phi\big(h^*(W^*), t\big)$ at a point $u \in N$, thus we obtain that

$$\frac{d}{dt}\sup_{u\in W^*}\tilde{E}\big(\Phi\big(h^*(u), t\big)\big) \leq -c < 0\ ,$$

uniformly in $t \geq 0$. (Here, condition (P.-S.) was used; the constant c may depend on our initial choice for the map h.) Replacing h^* by $\Phi(\cdot, t) \circ h^*$ and choosing t large, if necessary, we hence may assume that $\tilde{E}$ achieves its supremum on $h^*(W^*)$ in N. That is, we may estimate

$$\begin{aligned}\tilde{\beta}_k + \varepsilon = \tilde{\beta}_k^* + \varepsilon &\geq \sup_{u\in W^*}\tilde{E}\,(h^*(u)) - \sup_{\substack{u\in N,\\ \tilde{E}(u)\leq\tilde{\beta}_k+\varepsilon}}|\tilde{E}(u) - \tilde{E}(-u)| \\ &= \sup_{u\in W^*}\tilde{E}\,(h^*(u)) - \sup_{\substack{u\in N,\\ \tilde{E}(u)\leq\tilde{\beta}_k+\varepsilon}}|2\int_\Omega f u\,dx| \\ &\geq \tilde{\beta}_{k+1} - c\sup_{\substack{u\in N\\ \tilde{E}(u)\leq\tilde{\beta}_k+\varepsilon}}\|u\|_{L^2}\ .\end{aligned} \tag{7.4}$$

But for $u \in N$ with $\tilde{E}(u) \leq \tilde{\beta}_k + \varepsilon$, with constants $c > 0$ from (3°) and (6.4), (6.5) we obtain

$$\begin{aligned}
\|u\|_{L^2}^q &\le c\,\|u\|_{L^q}^q \le c\left(\frac{q-2}{2}\right)\int_\Omega G(u)\,dx + C \\
&\le c\int_\Omega \left(\frac{1}{2}g(u)u - G(u)\right)\,dx + C \\
&= c\left(\tilde{E}(u) - \frac{1}{2}\langle u, D\tilde{E}(u)\rangle + \frac{1}{2}\int_\Omega fu\,dx\right) + C \\
&\le c\,\tilde{E}(u) + c\left(1 + \|D\tilde{E}(u)\|\right)\|u\|_{H_0^{1,2}} + C \\
&\le c\,\tilde{E}(u) + c\left(1 + \|D\tilde{E}(u) - DE(u)\|\right)\|u\|_{H_0^{1,2}} + C \\
&\le c\,\tilde{\beta}_k + c\,\|u\|_{H_0^{1,2}} + C\ .
\end{aligned}$$

From this there follows

$$\begin{aligned}
\|u\|_{H_0^{1,2}}^2 &\le 2\,\tilde{E}(u) + 2\int_\Omega G(u)\,dx + C\|u\|_{L^2} \\
&\le c\,\tilde{\beta}_k + c\,\|u\|_{H_0^{1,2}} + C\ .
\end{aligned}$$

Together with the former estimate this shows that for such u there holds

$$\|u\|_{L^2} \le c\,\tilde{\beta}_k^{1/q} + C \le c\,\tilde{\beta}_k^{1/q}\ .$$

Inserting this estimate in (7.4) above yields the inequality

$$\tilde{\beta}_{k+1} \le \tilde{\beta}_k + c\,\tilde{\beta}_k^{1/q} = \tilde{\beta}_k\left(1 + c\,\tilde{\beta}_k^{\frac{1-q}{q}}\right)\ ,$$

for all $k \ge k_0$, with a uniform constant c. By iteration therefore

$$\begin{aligned}
\tilde{\beta}_{k_0+l} &\le \tilde{\beta}_{k_0}\prod_{k=k_0}^{k_0+l-1}\left(1 + c\,\tilde{\beta}_k^{\frac{1-q}{q}}\right) \\
&\le \tilde{\beta}_{k_0}\,exp\left(\sum_{k=k_0}^{k_0+l-1} ln\left(1 + c\,\tilde{\beta}_k^{\frac{1-q}{q}}\right)\right) \\
&\le \tilde{\beta}_{k_0}\,exp\left(c\sum_{k=k_0}^{k_0+l-1}\tilde{\beta}_k^{\frac{1-q}{q}}\right)\ .
\end{aligned}$$

But by (7.2), (7.3) we have

$$\tilde{\beta}_k \ge \beta_k \ge c\,k^{\frac{2p}{n(p-2)}-1}\ .$$

Since we assume that

$$\mu = \frac{1-q}{q}\cdot\left(\frac{2p}{n(p-2)} - 1\right) < -1\ ,$$

we can uniformly estimate

$$\tilde{\beta}_{k_0+l} \le \tilde{\beta}_{k_0}\ exp\left(c\sum_{k=k_0}^{k_0+l-1}\tilde{\beta}_k^{\frac{1-q}{q}}\right)$$
$$\le \tilde{\beta}_{k_0}\ exp\left(c\sum_{k=k_0}^{\infty}k^{\mu}\right) < \infty\ ,$$

for all $l \in \mathbb{N}$, which yields the desired contradiction. Thus, $\tilde{E}$ possesses an unbounded sequence of critical values and the proof is complete. □

Remark 7.3. (1°) In the special case $g(x,u) = u|u|^{p-2}$, by Theorem 7.2 for any $f \in L^2$ the problem

$$-\Delta u = u|u|^{p-2} + f \qquad \text{in } \Omega\ , \tag{7.5}$$
$$u = 0 \qquad \text{on } \partial\Omega\ , \tag{7.6}$$

admits infinitely many solutions, provided $2 < p < p^*$, where $p^* > 2$ is the largest root of the equation

$$\frac{2p^*}{n(p^*-2)} = 2 + \frac{1}{p^*-1}\ . \tag{7.7}$$

Observe that $p^* < 2^*$.

(2°) Recently, Bahri-Lions [1] have been able to improve the estimate for β_k as follows

$$\beta_k \ge c\ k^{\frac{2p}{n(p-2)}}\ ,$$

which yields the improved bounds

$$\frac{q}{q-1} < \frac{2p}{n(p-2)}$$

for p and q for which Theorem 7.2 is valid. In particular, for the model problem (7.5), (7.6) above, one obtains infinitely many solutions, provided $2 < p < \bar{p}$, where $2 < \bar{p} < 2^*$ solves the equation

$$\frac{2\bar{p}}{n(\bar{p}-2)} = 1 + \frac{1}{\bar{p}-1}\ .$$

(3°) Bahri [1] has shown that for any $p \in]2, 2^*[$ there is a dense open set of $f \in H^{-1}(\Omega)$ for which problem (7.5), (7.6) possesses infinitely many solutions.

Generally, it is expected that this should be true for *all* f (for instance in $L^2(\Omega)$), also for more general problems of type (6.1), (6.2) considered earlier. Indeed, this is true in the case of ordinary differential equtions – see for example Nehari [1], Struwe [2], or Turner [1] – and in the radially symmetric case, where different solutions of (6.1), (6.2) can be characterized by the nodal properties they possess; see Struwe [3], [4]. However, the proof of this conjecture in general remains open.

(4°) Applications of perturbation theory to Hamiltonian systems are given by Bahri-Berestycki [2], Rabinowitz [10], and – more recently – by Long [1].

(5°) Finally, Tanaka [1] has obtained similar perturbation results for nonlinear wave equations, as in Section I.6.

8. Linking

In the preceding chapter we have seen that symmetry is not essential for a variational problem to have "many" solutions. In this chapter we will describe another method for dealing with non-symmetric functionals. This method was introduced by Benci [1], Ni [1], and Rabinowitz [8]. Subsequently, it was generalized to a setting possibly involving also "indefinite" functionals by Benci-Rabinowitz [1]. It is based on the topological notion of "linking".

8.1 Definition. *Let S be a closed subset of a Banach space V, Q a submanifold of V with relative boundary ∂Q. We say S and ∂Q link if*
(1°) $S \cap \partial Q = \emptyset$, and
(2°) for any map $h \in C^0(V;V)$ such that $h|_{\partial Q} = id$ there holds $h(Q) \cap S \neq \emptyset$.
More generally, if S and Q are as above and if Γ is a subset of $C^0(V;V)$, then S and ∂Q will be said to link with respect to Γ, if (1°) holds and if (2°) is satisfied for any $h \in \Gamma$.

8.2 Example. Let $V = V_1 \oplus V_2$ be decomposed into closed subspaces V_1, V_2, where $\dim V_2 < \infty$. Let $S = V_1$, $Q = B_R(0;V_2)$ with relative boundary $\partial Q = \{u \in V_2 \ ; \ \|u\| = R\}$. Then S and ∂Q link.

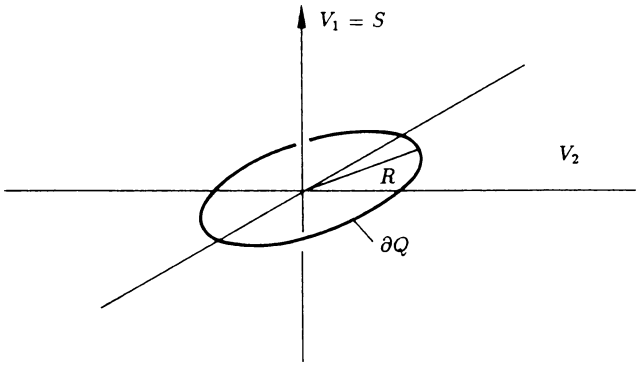

Fig. 8.1.

Proof. Let $\pi: V \to V_2$ be the (continuous) projection of V onto V_2, and let h be any continuous map such that $h|_{\partial Q} = id$. We have to show that $\pi\big(h(Q)\big) \ni 0$.

For $t \in [0,1]$, $u \in V_2$ define

$$h_t(u) = t\,\pi\big(h(u)\big) + (1-t)u \ .$$

Note that $h_t \in C^0(V_2; V_2)$ defines a homotopy of $h_0 = id$ with $h_1 = \pi \circ h$. Moreover, $h_t|_{\partial Q} = id$ for all t. Hence the topological degree $\deg(h_t, Q, 0)$ is well-defined for all t. By homotopy invariance and normalization of the degree (see for instance Deimling [1; Theorem 1.3.1]), we have

$$\deg(\pi \circ h, Q, 0) = \deg(id, Q, 0) = 1 \ .$$

Hence $0 \in \pi \circ h(Q)$, as was to be shown. □

8.3 Example. Let $V = V_1 \oplus V_2$ be decomposed into closed subspaces with $\dim V_2 < \infty$, and let $\underline{u} \in V_1$ with $\|\underline{u}\| = 1$ be given. Suppose $0 < \rho < R_1,\ 0 < R_2$ and let

$$S = \{u \in V_1 \ ; \ \|u\| = \rho\} \ ,$$
$$Q = \{s\underline{u} + u_2 \ ; \ 0 \le s \le R_1,\ u_2 \in V_2, \|u_2\| \le R_2\} \ ,$$

with relative boundary $\partial Q = \{s\underline{u} + u_2 \in Q \ ; \ s \in \{0, R_1\}$ or $\|u_2\| = R_2\}$. Then S and ∂Q link.

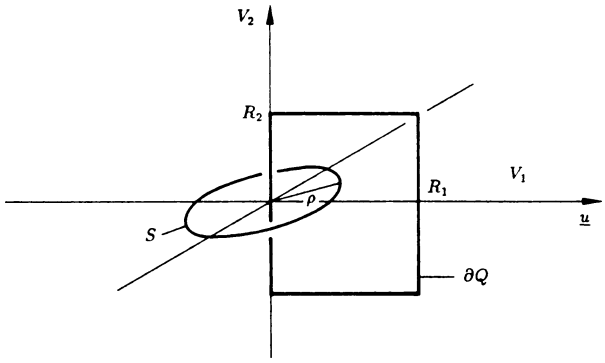

Fig. 8.2.

Proof. Let $\pi: V \to V_2$ denote the projection onto V_2, and let $h \in C^0(V; V)$ satisfy $h|_{\partial Q} = id$. We must show that there exists $u \in Q$ such that the relations $\|h(u)\| = \rho$ and $\pi\big(h(u)\big) = 0$ simultaneously hold. For $t \in [0,1]$, $s \in \mathbb{R}$, $u_2 \in V_2$ let

$$\overline{h}_t(s, u_2) = \big(t\|h(u) - \pi\big(h(u)\big)\| + (1-t)s - \rho,\ t\pi\big(h(u)\big) + (1-t)u_2\big) \ ,$$

where $u = s\underline{u} + u_2$. This defines a family of maps $\overline{h}_t: \mathbb{R} \times V_2 \to \mathbb{R} \times V_2$ depending continuously on $t \in [0,1]$. Moreover, if $u = s\underline{u} + u_2 \in \partial Q$, we have

$$\overline{h}_t(s, u_2) = \big(t\|u - u_2\| + (1-t)s - \rho, u_2\big) = (s - \rho, u_2) \neq 0$$

for all $t \in [0,1]$. Hence, if we identify Q with a subset of $\mathbb{R} \times V_2$ via the decomposition $u = s\underline{u} + u_2$, the topological degree $\deg(\overline{h}_t, Q, 0)$ is well-defined and by homotopy invariance

$$\deg(\overline{h}_1, Q, 0) = \deg(\overline{h}_0, Q, 0) = 1 \ ,$$

where $\overline{h}_0(s, u_2) = (s - \rho, u_2)$. Thus, there exists $u = s\underline{u} + u_2 \in Q$ such that $\overline{h}_1(u) = 0$, which is equivalent to

$$\pi\big(h(u)\big) = 0 \qquad \text{and} \qquad \|h(u)\| = \rho \ ,$$

as desired. □

8.4 Theorem. *Suppose $E \in C^1(V)$ satisfies (P.-S.). Consider a closed subset $S \subset V$ and a submanifold $Q \subset V$ with relative boundary ∂Q. Suppose*
(1°) S and ∂Q link,
(2°) $\alpha = \inf_{u \in S} E(u) > \sup_{u \in \partial Q} E(u) = \alpha_0$.
Let

$$\Gamma = \{h \in C^0(V; V) \ ; \ h|_{\partial Q} = id\} \ .$$

Then the number

$$\beta = \inf_{h \in \Gamma} \sup_{u \in Q} E\big(h(u)\big)$$

defines a critical value $\beta \geq \alpha$ of E.

Proof. Suppose by contradiction that $K_\beta = \emptyset$. For $\overline{\varepsilon} = \alpha - \alpha_0 > 0$, $N = \emptyset$ let $\varepsilon > 0$ and $\Phi\colon V \times [0,1] \to V$ be the pseudo-gradient flow constructed in Theorem 3.4. Note that by choice of $\overline{\varepsilon}$ there holds $\Phi(\cdot, t)|_{\partial Q} = id$ for all t. Let $h \in \Gamma$ such that $E\big(h(u)\big) < \beta + \varepsilon$ for all $u \in Q$. Define $h' = \Phi(\cdot, t) \circ h$. Then $h' \in \Gamma$ and

$$\sup_{u \in Q} E\big(h(u)\big) < \beta - \varepsilon$$

by Theorem 3.4(3°), contradicting the definition of β. □

Remark. By Example 8.3, letting $S = \partial B_\rho(0; V)$, $Q = \{t\underline{u} \ ; \ 0 \leq t \leq 1\}$ with relative boundary $\partial Q = \{0, \underline{u}\}$, $0 < \rho < \|\underline{u}\|$, Theorem 8.4 contains the mountain pass lemma Theorem 6.1 as a special case. In contrast to Theorem 6.1, the above Theorem 8.4 also allows higher-dimensional comparison sets Q, similar to Theorem 6.3 in the symmetric case.

Applications to Semilinear Elliptic Equations

We demonstrate this with yet another variant of Theorem 6.2, allowing for linear growth of g near $u = 0$. Conceivably, condition (1°) below may be further weakened. However, we will not pursue this.

8.5 Theorem. *Let Ω be a smooth, bounded domain in $\mathbb{R}^n, n \geq 3$, and let $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in $x \in \Omega$, differentiable in $u \in \mathbb{R}$ with derivative g_u, and with primitive G. Moreover, suppose that*
(1°) $g(x,0) = 0$, and

$$\frac{g(x,u)}{u} \geq g_u(x,0), \quad \text{for almost every } x \in \Omega, u \in \mathbb{R} \ ;$$

(2°) $\exists p < \frac{2n}{n-2}$, $C : \big|g_u(x,u)\big| \leq C\big(1 + |u|^{p-2}\big)$, for almost every $x \in \Omega, u \in \mathbb{R}$;
(3°) $\exists q > 2, R_0 : 0 < q\,G(x,u) \leq g(x,u)u$, for almost every $x \in \Omega$, if $|u| \geq R_0$.
Then the problem

$$-\Delta u = g(\cdot, u) \qquad \text{in } \Omega \tag{8.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{8.2}$$

admits a solution $u \not\equiv 0$.

Proof. For $u \in H_0^{1,2}(\Omega)$ let

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(x,u)\, dx \ .$$

By assumptions (2°), (3°) the functional E is of class C^2 and as in the proof of Theorem 6.2 satisfies (P.-S.) on $H_0^{1,2}(\Omega)$. It suffices to show that E admits a critical point $u \not\equiv 0$. Note that by assumption (1°) problem (8.1), (8.2) admits $u \equiv 0$ as a trivial solution.

Let φ_k denote the eigenfunctions of the linearized equation

$$\begin{aligned} -\Delta\varphi_k &= g_u(x,0)\varphi_k + \lambda_k\varphi_k \qquad && \text{in } \Omega \\ \varphi_k &= 0 && \text{on } \partial\Omega \end{aligned}$$

with eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$. Denote $k_0 = \min\{k \ ; \ \lambda_k > 0\}$ and let $V^+ = \operatorname{span}\{\varphi_k \ ; \ k \geq k_0\}$, $V^- = \operatorname{span}\{\varphi_1, \dots, \varphi_{k_0-1}\}$. Note that by (1°) we have

$$G(x,u) = \int_0^u g(x,v)\, dv \geq \frac{1}{2} g_u(x,0)u^2 \ .$$

Hence there holds

$$E(u) \leq \frac{1}{2} D^2E(0)(u,u) \leq 0, \ \text{for } u \in V^- \ , \tag{8.3}$$

while by definition of V^+ clearly we have

$$D^2E(0)(u,u) \geq \lambda_{k_o}\|u\|_{L^2}^2, \text{ for } u \in V^+ .$$

To strengthen the latter inequality note that by (2°) the function $g_u(x,0)$ is essentially bounded. Thus for sufficiently large k_1 we have

$$D^2E(0)(u,u) = \int_0 |\nabla u|^2\, dx - \int_\Omega g_u(x,0)u^2\, dx \geq \frac{1}{2}\|u\|_{H_0^{1,2}}^2$$

uniformly for $u \in \operatorname{span}\{\varphi_k \ ; \ k \geq k_1\}$. Since the complement of this space in V^+ has finite dimension, we conclude that there exists $\lambda > 0$ such that

$$D^2E(0)(u,u) \geq \lambda\|u\|_{H_0^{1,2}}^2 ,$$

uniformly for $u \in V^+$. But $E \in C^2\big(H_0^{1,2}(\Omega)\big)$; it follows that for sufficiently small $\rho > 0$ we have

$$\inf_{u \in S_\rho^+} E(u) \geq \frac{1}{2}D^2E(0)(u,u) - o\big(\|u\|_{H_0^{1,2}}^2\big) \geq \frac{\lambda\rho^2}{4} > 0 , \tag{8.4}$$

where $S_\rho^+ = \{u \in V^+ \ ; \ \|u\|_{H_0^{1,2}} = \rho\}$ and where $o(s)/s \to 0$ $(s \to 0)$.

By (3°) finally, as in the proof of Theorem 6.2 (see (6.4), (6.5)), for any finite dimensional subspace W we have

$$E(u) \to -\infty, \qquad \text{as } \|u\| \to \infty,\ u \in W .$$

Recalling (8.3), (8.4), we see that the assumptions of Theorem 8.4 are satisfied with $S = S_\rho^+$ and

$$Q = \{u^- + s\varphi_{k_0} \ ; \ u^- \in V^-,\ \|u^-\|_{H_0^{1,2}} \leq R,\ 0 \leq s \leq R\} ,$$

for sufficiently large $R > 0$. Thus E admits a critical point u with $E(u) \geq \frac{\lambda\rho^2}{4}$. The proof is complete. □

Further applications of Theorem 8.4 to semilinear elliptic boundary value problems are given in the survey notes by De Figueireido [1] or Rabinowitz [11; p. 25 ff.]. In particular, Theorem 8.4 offers a simple and unified approach to asymptotically linear equations, possibly "resonant" at $u = 0$ or at infinity, as in Ahmad-Lazer-Paul [1], Amann [4], Amann-Zehnder [1]. (See for instance Rabinowitz [11; Theorem 4.12], or Bartolo-Benci-Fortunato [1] and the references cited therein. See also Chang [1; p. 708].)

Applications to Hamiltonian Systems

A more refined application of linking is given in the next theorem due to Hofer and Zehnder [1], prompted by the work of Viterbo [1]. Once again we deal with Hamiltonian systems

$$\dot{x} = \mathcal{J}\nabla H(x) \ , \tag{8.5}$$

where H is a given smooth Hamiltonian and $\mathcal{J}$ is the skew-symmetric matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -id \\ id & 0 \end{pmatrix}$$

on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

8.6 Theorem. *Let $H \in C^2(\mathbb{R}^{2n}; \mathbb{R})$. Suppose that 1 is a regular value of H and $S = S_1 = H^{-1}(1)$ is compact and connected. Then for any $\delta > 0$ there is a number $\beta \in]1-\delta, 1+\delta[$ such that $S_\beta = H^{-1}(\beta)$ carries a periodic solution of the Hamiltonian system (8.5).*

(Note that by the implicit function theorem and compactness of S there exists a number $\delta_0 > 0$ such that for any $\beta \in]1-\delta_0, 1+\delta_0[$ the hypersurface S_β is of class C^2, compact, and diffeomorphic to S.)

Remarks. By Theorem 8.6, in order to obtain a periodic solution of (8.5) on the *fixed* surface $S = S_1$, it would suffice to obtain a-priori bounds (in terms of the action integral) for periodic solutions to (8.5) on surfaces S_β near S. Benci-Hofer-Rabinowitz [1] have shown that this is indeed possible, provided certain geometric conditions are satisfied, including, for instance, the condition that S be strictly star-shaped with respect to the origin. However, also energy surfaces of Hamiltonians that are only convex either in the position or in the momentum variables are allowed.

The existence of periodic solutions to (8.5) on S likewise can be derived if S is of "contact type". This notion, introduced by Weinstein [3], allows one to give an intrinsic interpretation of the existence results by Rabinowitz [5] and Weinstein [2] for closed trajectories of Hamiltonian systems on convex or strictly star-shaped energy hypersurfaces. See Hofer-Zehnder [1], or Zehnder [2] for details.

Proof of Theorem 8.6. Observe that, as remarked earlier in the proof of Theorem I.6.5, the property that a level hypersurface $S_\beta = H^{-1}(\beta)$ carries a periodic solution of (8.5) is independent of the particular Hamiltonian H having S_β as a level surface. We now use this freedom to redefine H suitably. For $0 < \delta < \delta_0$ let $U = H^{-1}\big([1-\delta, 1+\delta]\big) \simeq S \times [-1, 1]$. Then $\mathbb{R}^{2n} \setminus U$ has two components. Indeed, by Alexander duality

$$\tilde{H}_0(\mathbb{R}^{2n} \setminus U\ ;\ \mathbb{Z}) \simeq \overline{H}^{2n-1}(U\ ;\ \mathbb{Z}) \simeq \overline{H}^{2n-1}(S\ ;\ \mathbb{Z}) \simeq \mathbb{Z}\ ,$$

in Spanier's [1] notation. Denote by A the unbounded component of $\mathbb{R}^{2n} \setminus U$ and by B the bounded component. We may assume $0 \in B$. Also let $\gamma = \operatorname{diam} U > 0$. Fix numbers $r, b > 0$ such that

$$\begin{aligned} \gamma < & r < 2\gamma \\ \frac{3}{2}\pi r^2 < & b < 2\pi r^2 \end{aligned}$$

and choose a smooth function $f\colon]-\delta_0, \delta_0[\to \mathbb{R}$ such that $f|_{]-\delta_0,-\delta]} = 0$, $f|_{[\delta,\delta_0[} = b$, and such that $f'(s) > 0$ for $-\delta < s < \delta$. Also let $g\colon \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$\begin{aligned} &g(s) = b \ \text{ for } s \le r,\ g(s) \ge \frac{3}{2}\pi s^2 \text{ for } s > r,\ g(s) = \frac{3}{2}\pi s^2 \text{ for large } s\ , \\ &\text{and } 0 < g'(s) \le 3\pi s \text{ for } s > r\ . \end{aligned} \tag{8.6}$$

Then define

$$\tilde{H}(x) = \begin{cases} 0, & \text{if } x \in B, \\ f(s), & \text{if } H(x) = 1 + s,\ -\delta \le s \le \delta, \\ b, & \text{if } x \in A,\ |x| \le r, \\ g(|x|), & \text{if } |x| > r\ . \end{cases}$$

Note that for $x \in \mathbb{R}^{2n}$ we can estimate

$$-b + \frac{3}{2}\pi|x|^2 \le \tilde{H}(x) \le \frac{3}{2}\pi|x|^2 + b\ . \tag{8.7}$$

Now we look for solutions of (8.5) of period 1, with $\tilde{H}$ replacing H. Such solutions uniquely correspond to the critical points of the functional

$$E(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \mathcal{J}x\rangle\ dt - \int_0^1 \tilde{H}(x)\ dt$$

on the space of 1-periodic (C^1-)maps $x\colon \mathbb{R} \to \mathbb{R}^{2n}$.

By the following lemma we are able to distinguish a periodic solution on a level hypersurface S_β.

8.7 Lemma. *Suppose x is a 1-periodic C^1-solution of (8.5), with $\tilde{H}$ replacing H, and which satisfies $E(x) > 0$. Then $x(t) \in S_\beta$ for some $\beta \in [1-\delta, 1+\delta]$, for all t.*

Proof. Since $\tilde{H} \ge 0$, any constant solution x of (8.5) satisfies $E(x) \le 0$. Assume x is non-constant with $|x(t_0)| > r$. Then $g\big(|x(t_0)|\big) = \tilde{H}\big(x(t_0)\big) = \tilde{H}\big(x(t)\big) = g\big(|x(t)|\big)$ also for t close to t_0 and by (8.6) it follows that $|x(t)| \equiv |x(t_0)| = s_0$. In particular, x satisfies

$$\dot{x} = \mathcal{J}\frac{g'(s_0)\, x}{s_0}$$

and we compute

$$\begin{aligned} E(x) &= \frac{1}{2} g'(s_0)\; s_0 - \tilde{H}\big(x(t_0)\big) \\ &\leq \frac{3}{2}\pi s_0^2 - \frac{3}{2}\pi s_0^2 = 0 \ , \end{aligned}$$

by definition of $\tilde{H}$. The claim now follows. □

In order to exhibit a 1-periodic solution x of (8.5) with $E(x) > 0$, a variational argument involving linking will be applied.

Denote by $V = H^{1/2,2}(S^1\ ;\ \mathbb{R}^{2n})$ the Hilbert space of all 1-periodic functions

$$x(t) = \sum_{k\in\mathbb{Z}} \exp\,(2\pi k t\mathcal{J})\; x_k \in L^2\big([0,1]\ ;\ \mathbb{R}^{2n}\big)\ , \tag{8.8}$$

with Fourier coefficients $x_k \in \mathbb{R}^{2n}$ satisfying

$$\|x\|^2 = 2\pi\left(\sum_{k\in\mathbb{Z}} |k|\ |x_k|^2\right) + |x_0|^2 < \infty\ .$$

The inner product on V is given by

$$(x,y)_V := 2\pi\left(\sum_{k\in\mathbb{Z}} |k|\langle x_k, y_k\rangle\right) + \langle x_0, y_0\rangle$$

inducing the norm $\|\cdot\|$ above.

Split $V = V^- \oplus V^0 \oplus V^+$ orthogonally, where

$$\begin{aligned} V^- &= \{x \in V\ ;\ x_k = 0 \text{ for } k \geq 0\}\ , \\ V^0 &= \{x \in V\ ;\ x_k = 0 \text{ for } k \neq 0\}\ , \\ V^+ &= \{x \in V\ ;\ x_k = 0 \text{ for } k \leq 0\} \end{aligned}$$

with reference to the Fourier decomposition (8.8) of an element $x \in V$. Also denote P^-, P^0 and P^+ the corresponding projections. Thus, any $x \in V$ may be uniquely expressed $x = x^- + x^0 + x^+$, where $x^- = P^- x$, etc.

Note that the self-adjoint operator

$$Lx = -\mathcal{J}\dot{x}$$

has eigenspaces

$$V_k = \{\exp(2\pi k t\mathcal{J})x_k\ ;\ x_k \in \mathbb{R}^{2n}\}$$

with eigenvalues $2\pi k,\ k \in \mathbb{Z}$. Moreover, $\dim V_k = 2n$, for all k. Hence the related quadratic form

$$A(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \mathcal{J}x\rangle \, dt \quad \begin{cases} < 0, & \text{for } x \in V^-, \\ = 0, & \text{for } x \in V^0, \\ > 0, & \text{for } x \in V^+. \end{cases}$$

In fact, using the projections $P^{\pm}$, A can be written

$$A(x) = \frac{1}{2}\big((-P^- + P^+)x, x\big)_V \ .$$

Note that since $\tilde{H}$ is of class C^2 and behaves like $|x|^2$ for large $|x|$, its mean

$$G(x) = \int_0^1 \tilde{H}(x) \, dt$$

is of class C^2 on $L^2(S^1; \mathbb{R}^{2n})$. Hence also $E \in C^2(H^{1/2,2}(S^1))$. Restricting G to V, from compactness of the embedding $H^{1/2,2}(S^1) \hookrightarrow L^2(S^1)$ we also deduce that DG is a completely continuous map from V into its dual.

8.8 Lemma. *There exist numbers $\alpha > 0$, $\rho \in]0,1[$ such that $E(x) \geq \alpha$ for $x \in V^+$, $\|x\| = \rho$.*

Proof. $\nabla\tilde{H}(0) = 0$, $\nabla^2\tilde{H}(0) = 0$ implies that $DG(0) = 0$, $D^2G(0) = 0$. Hence $D^2E(0) = -P^- + P^+$. Since $E \in C^2$, $E(0) = 0$, $DE(0) = 0$, the lemma follows. □

Now define

$$Q = \{x = x^- + x^0 + s\, e \ \in V \ ; \ \|x^- + x^0\| \leq R, \text{ and } 0 \leq s \leq R\} \ ,$$

with $R > 1$ to be determined, and with a unit vector

$$e = \frac{1}{\sqrt{2\pi}} \exp\,(2\pi t\mathcal{J})a \in V^+, \qquad |a| = 1 \ .$$

Denote ∂Q the relative boundary of Q in $V^- \oplus V^0 \oplus \mathbb{R}\cdot e$; that is,

$$\partial Q = \big\{x^- + x^0 + s\, e \in Q \ ; \ \|x^- + x^0\| = R \ \text{ or } \ s \in \{0, R\}\big\} \ .$$

8.9 Lemma. *If R is sufficiently large, then $E|_{\partial Q} \leq 0$.*

Proof. Note that (8.7) implies that

$$G(x) = \int_0^1 \tilde{H}(x) \, dt \geq -b + \frac{3}{2}\pi\|x\|_{L^2}^2 \ .$$

Therefore, for $x = x^- + x^0 + se$ with $s = R$ or $\|x^- + x^0\|^2 = \|x^-\|^2 + \|x^0\|_{L^2}^2 = R$, we obtain

$$\begin{aligned} E(x) &= \frac{1}{2}\big((-P^- + P^+)x, x\big)_V - G(x) \\ &\le -\frac{1}{2}\|x^-\|^2 + \frac{1}{2}s^2 + b - \frac{3}{2}\pi\big(\|x^-\|_{L^2}^2 + \|x^0\|_{L^2}^2 + \frac{s^2}{2\pi}\big) \\ &\le -\frac{1}{2}\|x^-\|^2 - \frac{1}{4}s^2 - \frac{3}{2}\pi\|x^0\|_{L^2}^2 + b \\ &\le 0\ , \end{aligned}$$

if $R > 0$ is sufficiently large. Moreover, $\tilde{H} \ge 0$ implies $G \ge 0$ and therefore $E \le 0$ on $V^- \oplus V^0$, that is at $s = 0$, which concludes the proof. □

Fix $0 < \rho < R$ as in Lemmas 8.8, 8.9. Denote $S_\rho^+ = \{x \in V^+\ ;\ \|x\| = \rho\}$. Define a class Γ of maps $V \to V$ as follows:

Γ is the class of maps $h \in C^0(V;V)$ such that h is homotopic to the identity through a family of maps $h_t = L_t + K_t,\ 0 \le t \le T$, where $L_0 = id,\ K_0 = 0$, and where for each t there holds:

$$L_t = (L_t^-, L_t^0, L_t^+)\colon V^- \oplus V^0 \oplus V^+ \to V^- \oplus V^0 \oplus V^+$$

is a linear isomorphism preserving sub-spaces, K_t is compact and $h_t(\partial Q) \cap S_\rho^+ = \emptyset$, for each t.

Following Benci-Rabinowitz [1], we now establish

8.10 Lemma. *∂Q and S_ρ^+ link with respect to Γ.*

Proof. Choose $h \in \Gamma$. We must show that $h(Q) \cap S_\rho^+ \ne \emptyset$, or equivalently that the equations

$$\big((P^- + P^0) \circ h\big)(x) = 0, \qquad \|h(x)\| = \rho \tag{8.9}$$

are satisfied for some $x \in Q$.

Using a degree argument as in Example 8.3, we establish (8.9) for every h_t in the family defining $h \in \Gamma$. Consider $Q \subset V^- \oplus V^0 \oplus \mathbb{R} \cdot e =: W$ and represent $x = x^- + x^0 + se$. Applying L_t^{-1} to (8.9), these equations become

$$\begin{aligned} x^- + x^0 + (P^- + P^0)L_t^{-1}K_t(x) &= 0\ , \\ \|h_t(x)\| &= \rho\ . \end{aligned}$$

Note that $k_t := (P^- + P^0)L_t^{-1}K_t\colon W \to V^- \oplus V^0$ is compact. Now define a map $T_t\colon W \to W$ by letting

$$\begin{aligned} T_t(x) = T_t(x^- + x^0 + se) &= x^- + x^0 + k_t(x) + \|h_t(x)\|e \\ &= x + k_t(x) + \big(\|h_t(x)\| - s\big)e\ . \end{aligned}$$

Observe that T_t is of the form $id + compact$. Moreover, the condition $h_t(\partial Q) \cap S_\rho^+ = \emptyset$ translates into the condition $T_t(x) \neq \rho e$ for $x \in \partial Q$. Hence the Leray-Schauder degree (see for instance Deimling [1; 2.8.3, 2.8.4]) of T_t on Q with respect to ρe is well-defined and, in fact,

$$\deg(T_t, Q, \rho e) = \deg(T_0, Q, \rho e) = 1 \ ,$$

as desired. □

Let ∇E denote the gradient of E with respect to the scalar product in V. Since ∇E has linear growth, the gradient flow $\Phi\colon V \times [0, \infty[\to V$ given by

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(x,t) &= -\nabla E\big(\Phi(x,t)\big) \\ \Phi(x,0) &= x \end{aligned}$$

exists globally. Note that E is non-increasing along flow-lines with

$$\frac{d}{dt}E\big(\Phi(x,t)\big) = -\left\|\nabla E\big(\Phi(x,t)\big)\right\|^2 \ . \tag{8.10}$$

8.11 Lemma. *$\Phi(\cdot, T) \in \Gamma$ for any $T \geq 0$.*

Proof. Clearly $h_t = \Phi(\cdot, t)$ for $0 \leq t \leq T$ is a homotopy of $\Phi(\cdot, T)$ to the identity; moreover by (8.10)

$$E\big(\Phi(x,t)\big) \leq E(x) \leq 0 \qquad \text{for } x \in \partial Q \ .$$

By Lemmata 8.8 and 8.9 therefore $h_t(\partial Q) \cap S_\rho^+ = \emptyset$ for all t. Finally, observe that

$$\nabla E(x) = -x^- + x^+ + \nabla G(x) \ , \tag{8.11}$$

where ∇G is compact. Thus the desired form of $\Phi(x,t)$ may be read off the variation of constant formula

$$\begin{aligned} \Phi(x,t) &= e^t x^- + x^0 + e^{-t}x^+ + \int_0^t \big(e^{t-s}P^- + P^0 + e^{-(t-s)}P^+\big)\nabla G\big(\Phi(x,s)\big)\, ds \\ &=: L_t^- x^- + L_t^0 x^0 + L_t^+ x^+ + K_t(x) \ . \end{aligned}$$

□

It remains to verify the Palais-Smale condition for E.

8.12 Lemma. *E satisfies (P.-S.) on V.*

Proof. Let (x_m) be a (P.-S.)sequence for E. By (8.11) and Proposition 2.2 it suffices to show that (x_m) is bounded in V. Suppose by contradiction that $\|x_m\| \to \infty$. Normalize $z_m = \frac{x_m}{\|x_m\|}$ and note that

$$(-P^- + P^+)z_m - \frac{\nabla G(x_m)}{\|x_m\|} \to 0 \ . \tag{8.12}$$

We claim that the sequence $y_m := \frac{\nabla G(x_m)}{\|x_m\|}$ is precompact. Indeed, by (8.12), the sequence (y_m) is bounded and we may assume that $y_m \to y$ weakly in $H^{1/2,2}$ and strongly in L^2. Moreover, by (8.6), for $\phi \in H^{1/2,2}(S^1)$ we may estimate

$$\left| \left(\frac{\nabla G(x_m)}{\|x_m\|}, \phi \right)_V \right| = \left| \int_0^1 \frac{\langle \nabla \tilde{H}(x_m), \phi \rangle}{\|x_m\|} \, dt \right| \le c \, \left(1 + \left\| \frac{x_m}{\|x_m\|} \right\|_{L^2} \right) \|\phi\|_{L^2} \ .$$

Choosing $\phi = \phi_m = y_m - y$ above, it then follows that $y_m \to y$ strongly in $H^{1/2,2}$ as $m \to \infty$. Hence, from (8.12) it follows that $z_m \to z$ in $H^{1/2,2}$ and almost everywhere in $[0,1]$. Since $\nabla \tilde{H}$ grows linearly, passing to the limit in the expression

$$\left\langle \frac{\phi}{\|x_m\|}, DE(x_m) \right\rangle = \int_0^1 \langle -\mathcal{J}\dot{z}_m, \phi \rangle - \frac{\big\langle \nabla \tilde{H}\big(\|x_m\| z_m\big), \phi \big\rangle}{\|x_m\|} \, dt$$

is allowed by Vitali's convergence theorem. From (8.6) we thus infer that z satisfies the equation

$$\dot{z} = 3\pi \mathcal{J} \, z \ .$$

However, 3π does not belong to the spectrum of $L = -\mathcal{J}\frac{d}{dt}$. Hence $z \equiv 0$. But since $z_m \to z$, and $\|z_m\| = 1$, this is impossible. Therefore the original sequence (x_m) must be bounded. □

In order to conclude the proof of Theorem 8.6 we can now repeat the argument of Theorem 8.4. Define

$$\beta = \inf_{t \ge 0} \sup_{x \in Q} E\big(\Phi(x,t)\big) \ .$$

From Lemmata 8.10 and 8.11 it follows that $\Phi(Q,t) \cap S^+_\rho \ne \emptyset$ for all $t \ge 0$. Hence $\beta \ge \alpha > 0$. Suppose by contradiction that β is a regular value for E. By (P.-S.) there exists $\delta > 0$ such that

$$\|\nabla E(x)\| \ge \delta$$

for all x such that $|E(x) - \beta| < \delta$. Choose $t_0 > 0$ such that

$$\sup_{x \in Q} E\big(\Phi(x,t_0)\big) < \beta + \delta \ .$$

Then by definition of β, choice of δ, and (8.10), for $t \geq t_0$ the number

$$\sup_{x \in Q} E\big(\Phi(x,t)\big)$$

is achieved only at points where $\Big\|\nabla E\big(\Phi(x,t)\big)\Big\| \geq \delta$.

Hence by (8.10)

$$\frac{d}{dt}\left(\sup_{x \in Q} E\big(\Phi(x,t)\big)\right) \leq -\delta^2 \ ,$$

which gives a contradiction to the definition of β after time $T \geq t_0 + \frac{1}{\delta}$. The proof is complete. □

In the next section we will see that by a slight refinement of the above linking argument one can obtain periodic solutions of (8.5) not only with energy arbitrarily close to a given value β_0, but, in fact, for almost every energy level in a suitable neighborhood of β_0 – a vast abundance of periodic solutions!

9. Parameter Dependence

The existence result for periodic solutions of Hamiltonian systems given in the previous section can be considerably improved, if combined with the ideas presented in Section I.7.

In fact, we have the following result of Struwe [21].

9.1 Theorem. *Let $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and suppose that 1 is a regular value of H and the energy level surface $S = S_1 = \{x; H(x) = 1\}$ is compact and connected. Then there is a number $\delta_0 > 0$ such that for almost every $\beta \in]\, 1 - \delta_0, 1 + \delta_0\, [$ there exists a periodic solution x of (8.5) with $H(x(t)) \equiv \beta$.*

In fact, $\delta_0 > 0$ may be any number as determined in the remark following Theorem 8.6 such that for $\beta \in]\, 1 - \delta_0, 1 + \delta_0\, [$ the level surface $S_\beta = H^{-1}(\beta)$ is C^2-diffeomorphic to S.

For the proof of Theorem 9.1 we may modify the Hamiltonian H as in the preceding section while preserving the level surfaces S_β, $|1 - \beta| < \delta_0$. However, now we shift $\tilde{H}$, as follows. Fix any number $\beta_0 \in]\, 1 - \delta_0, 1 + \delta_0\, [$ and let $0 < \delta < \big(\delta_0 - |1 - \beta_0|\big)/3$. If suffices to show that (8.5) has a periodic solution with energy β for almost every $\beta \in]\, \beta_0 - \delta\ , \beta_0 + \delta\, [=: I_0$.

Let $U = H^{-1}\big([1 - \delta_0 + \delta, 1 + \delta_0 - \delta]\big) \cong S \times [-1, 1]$, $\mathbb{R}^{2n} \setminus U = A \cup B$, $\gamma = \operatorname{diam} U$, $\gamma < r < 2\gamma$, $\frac{3}{2}\pi r^2 < b < 2\pi r^2$ as in Section 8 above and choose smooth functions f and g as in that section such that $f(s) = 0$ for $s \leq -\delta$, $f(s) = b$ for $s \geq \delta$, $f'(s) > 0$ for $|s| < \delta$, and where g satisfies (8.6). For $\beta \in I_0$, $m \in \mathbb{N}$, then define

$$\tilde{H}_{\beta,m}(x) = \begin{cases} 0\ , & \text{if } x \in B\ , \\ f\big(m(H(x)-\beta)\big)\ , & \text{if } x \in U\ , \\ b\ , & \text{if } x \in A, |x| \le r\ , \\ g\big(|x|\big)\ , & \text{if } |x| > r\ , \end{cases}$$

satisfying (8.7), and for $x \in V = H^{1/2,2}(S^1;\mathbb{R}^{2n})$ let

$$\begin{aligned} E_{\beta,m}(x) &= \frac{1}{2}\int_0^1 \langle \dot{x}, Jx\rangle\ dt - \int_0^1 \tilde{H}_{\beta,m}(x)\ dt \\ &= A(x) - G_{\beta,m}(x)\ . \end{aligned}$$

Then, as in Section 8, the functional $E_{\beta,m}$ is of class C^2 and satisfies (P.-S.)on V. Moreover, critical points $x \in V$ of $E_{\beta,m}$ with $E_{\beta,m}(x) > 0$ correspond to periodic solutions y of (8.5) with energy $|H(y) - \beta| < \delta/m$; compare Lemma 8.7. More precisely, we have

9.2 Lemma. *Let $x \in V$ be a critical point of $E_{\beta,m}$ with $E_{\beta,m}(x) > 0$. Then $H\big(x(t)\big) \equiv h \in \mathbb{R}$ with $|h - \beta| < \delta/m$, and letting*

$$T(x) = mf'\bigg(m\big(H(x)-\beta\big)\bigg) > 0\ ,$$

the function $y(t) = x\big(t/T(x)\big)$ is a $T(x)$-periodic solution of (8.5).

Proof. The first assertion follows just as in the proof of Lemma 8.9. The second assertion is immediate from the equation

$$\begin{aligned} \dot{x} = \mathcal{J}\nabla\tilde{H}_{\beta,m}(x) &= mf'\bigg(m\big(H(x)-\beta\big)\bigg)\mathcal{J}\nabla H(x) \\ &= T(x)\mathcal{J}\nabla H(x)\ . \end{aligned} \tag{9.1}$$

□

Moreover, we have (compare Lemma 8.8):

9.3 Lemma. *There exist numbers $\alpha > 0, \rho > 0$ such that $E_{\beta,m}(x) \ge \alpha$ for $x \in S^+_\rho = \big\{x \in V^+; \|x\| = \rho\big\}$, uniformly in $m \in \mathbb{N}$, $\beta \in I_0$.*

Proof. Let $r_0 > 0$ be such that $B_{r_0}(0;\mathbb{R}^{2n}) \subset B$. By Tchebychev's inequality, we have

$$\mathcal{L}^1\bigg(\big\{t; |x(t)| > r_0\big\}\bigg) \le r_0^{-2}\int_0^1 |x(t)|^2\ dt \le C\|x\|^2$$

for $x \in V$. There exists a constant C_0 such that

$$\tilde{H}_{\beta,m}(x) \le C_0|x|^2 \qquad \text{for all } x\ ,$$

uniformly in $m \in \mathbb{N}$, $\beta \in I_0$. Since $V \hookrightarrow L^4\big([0,1];\mathbb{R}^{2n}\big)$, moreover, from Hölders inequality we obtain

$$G_{\beta,m}(x) = \int_0^1 \tilde{H}_{\beta,m}\big(x(t)\big)\ dt \le C_0 \int_{\{t;|x(t)|\ge r_0\}} |x(t)|^2\ dt$$
$$\le C_0 \mathcal{L}^1\Big(\big\{t;|x(t)|\ge r_0\big\}\Big)^{1/2} \Big(\int_0^1 |x(t)|^4\ dt\Big)^{1/2} \le C\|x\|^3\ .$$

On the other hand, for $x \in V^+$ there holds

$$A(x) = \frac{1}{2}\|x\|^2\ ,$$

and the claim follows. □

In addition, there exists a uniform number $R > \rho$ such that

$$E_{\beta,m}|_{\partial Q} \le 0$$

for all $\beta \in I_0$, $m \in \mathbb{N}$, where Q is defined as in Section 8; compare Lemma 8.9.

Finally, the gradient flow $\Phi_{\beta,m}(\cdot,t)$ for $E_{\beta,m}$ for any $t \ge 0$ is a member of Γ, for each $\beta \in I_0$, $m \in \mathbb{N}$.

Hence, as in Section 8, for each $\beta \in I_0$, $m \in \mathbb{N}$ there exists a periodic solution $x_{\beta,m}$ of (9.1) with

$$E_{\beta,m}(x_{\beta,m}) = \inf_{h\in\Gamma} \sup_{x\in Q} E_{\beta,m}\big(h(x)\big) =: \nu_m(\beta).$$

We now use variations of β to obtain a-priori bounds on $T(x_{\beta,m})$ that allow passing to the limit $m \to \infty$ in (9.1) for a suitable sub-sequence and suitable values of β.

Observe that, since $f' \ge 0$, for any fixed $x \in V$, $m \in \mathbb{N}$ the map $\beta \mapsto E_{\beta,m}(x)$ is non-decreasing in $\beta \in I_0$; in fact, for any $x \in V$ we have

$$\frac{\partial}{\partial\beta} E_{\beta,m}(x) = m \int_0^1 f'\Big(m\big(H(x(t)) - \beta\big)\Big)\ dt\ .$$

In particular, if $x \in V$is critical for $E_{\beta,m}$ with $E_{\beta,m}(x) > 0$, by Lemma 9.2 this yields

$$\frac{\partial}{\partial\beta} E_{\beta,m}(x) = T(x)\ .$$

As a consequence of the monotonicity of $E_{\beta,m}$, clearly also the map $\beta \mapsto \nu_m(\beta)$ is non-decreasing for any $m \in \mathbb{N}$. Hence this map is almost everywhere differentiable for any m with $\frac{\partial}{\partial\beta}\nu_m \ge 0$, and there holds

$$\int_{-\delta}^{\delta} \frac{\partial}{\partial\beta}\nu_m d\beta \le \sup_{m,\beta} \nu_m(\beta) - \alpha \le \sup_{m,\beta}\Big\{\sup_{x\in Q} E_{\beta,m}(x)\Big\} - \alpha = C < \infty\ ,$$

uniformly in $m \in \mathbb{N}$, with α as determined in Lemma 9.3.

But then also

$$\liminf_{m\to\infty}\big(\frac{\partial}{\partial\beta}\nu_m\big) \in L^1\big([-\delta,\delta]\big),$$

and by Fatou's lemma we obtain

$$\int_{-\delta}^{\delta} \liminf_{m\to\infty}\big(\frac{\partial}{\partial\beta}\nu_m\big)\ d\beta \le \liminf_{m\to\infty}\int_{-\delta}^{\delta}\frac{\partial}{\partial\beta}\nu_m \le C\ .$$

In particular, $\liminf_{m\to\infty}\frac{\partial}{\partial\beta}\nu_m(\beta) < \infty$ for almost every $\beta \in I_0$. Fix such a β and let $\Lambda \subset \mathbb{N}$ be a subsequence such that

$$\frac{\partial}{\partial\beta}\nu_m(\beta) \to \liminf_{m\to\infty}\frac{\partial}{\partial\beta}\nu_m(\beta) = C_\beta < \infty$$

as $m \to \infty$, $m \in \Lambda$. We may assume $\frac{\partial}{\partial\beta}\nu_m(\beta) \le C_\beta + 1$ for all $m \in \Lambda$.

We claim:

9.4 Lemma. *For any $m \in \Lambda$ there exists a critical point x_m of $E_{\beta,m}$ such that $E_{\beta,m}(x_m) = \nu_m(\beta)$ and $T(x_m) = \frac{\partial}{\partial\beta}E_{\beta,m}(x_m) \le C_\beta + 4$.*

Proof: Choose a sequence $\beta_k \searrow \beta$. We claim there exists a (P.-S.)-sequence (x_m^k) for $E_{\beta,m}$ such that $\nabla E_{\beta,m}(x_m^k) \to 0$ as $k \to \infty$ and such that

$$\begin{aligned}\nu_m(\beta) - 2(\beta_k - \beta) \le E_{\beta,m}(x_m^k) \le E_{\beta_k,m}(x_m^k)\\ \le \nu_m(\beta_k) + (\beta_k - \beta) \le \nu_m(\beta) + (C_\beta + 2)(\beta_k - \beta)\end{aligned} \tag{9.2}$$

for large k. This will imply the assertion of the lemma: By (P.-S.), a subsequence (x_m^k) as $k \to \infty$ will accumulate at a critical point x_m of $E_{\beta,m}$, satisfying $E_{\beta,m}(x_m) = \nu_m(\beta)$ and

$$\begin{aligned}C_\beta + 4 &\ge \liminf_{k\to\infty}\frac{E_{\beta_k,m}(x_m^k) - E_{\beta,m}(x_m^k)}{\beta_k - \beta}\\ &= \liminf_{k\to\infty}\int_0^1\int_\beta^{\beta_k}\frac{mf'\big(m(H(x_m^k) - \beta')\big)}{\beta_k - \beta}d\beta'\ dt\\ &= \int_0^1 mf'\big(m(H(x_m) - \beta)\big)\ dt = T(x_m)\ .\end{aligned}$$

Negating the above claim, suppose there is no (P.-S.)-sequence for $E_{\beta,m}$ satisfying (9.2). Then there exists $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $x \in V$ satisfying (9.2) for some $k \ge k_0$ there holds

$$\|\nabla E_{\beta,m}(x)\|^2 \ge 2\varepsilon\ . \tag{9.3}$$

We may assume $k_0 = 1$. Let $0 \le \varphi \le 1$ be a Lipschitz continuous function such that $\varphi(s) = 0$ for $s \le 0$, $\varphi(s) = 1$ for $s \ge 1$ and define a family of vector fields e_k by letting

$$e_k(x) = -\Big(\big(1-\varphi_k(x)\big)\nabla E_{\beta_k,m}(x) + \varphi_k(x)\nabla E_{\beta,m}(x)\Big)$$
$$= x^- - x^+ + \Big(\big(1-\varphi_k(x)\big)\nabla G_{\beta_k,m}(x) + \varphi_k(x)\nabla G_{\beta,m}(x)\Big)\ ,$$

where

$$\varphi_k(x) = \varphi\left(\frac{E_{\beta,m}(x) - \big(\nu_m(\beta) - 2(\beta_k-\beta)\big)}{\beta_k-\beta}\right)\ .$$

Let $\Phi_k\colon V\times[\,0,\infty[\,\to V$ be the corresponding flows, satisfying

$$\frac{\partial}{\partial t}\Phi_k(x,t) = e_k\big(\Phi_k(x,t)\big)\ ,$$
$$\Phi_k(x,0) = x$$

for all $x\in V, t\geq 0$. We claim that $\Phi_k(\cdot,t)\in\Gamma$ for any $t\geq 0$.

Indeed, we may assume $\nu_m(\beta)\geq 2(\beta_k-\beta)$ for all k. Thus, if $E_{\beta_k,m}(x)\leq 0$ for some $x\in V$, it follows that $\varphi_k(x)=0$, $e_k(x) = -\nabla E_{\beta_k,m}(x)$, and hence $E_{\beta_k,m}\big(\Phi_k(x,t)\big)\leq 0$ for all $t\geq 0$. Since this applies, in particular, to any point $x\in\partial Q$, it follows that

$$\Phi_k(\partial Q,t)\cap S_\rho^+ = \emptyset$$

for all $t\geq 0$. The remaining properties defining the class Γ are verified as in the proof of Lemma 8.11.

Choose $h\in\Gamma$ such that

$$\sup_{x\in Q} E_{\beta_k,m}\big(h(x)\big) \leq \nu_m(\beta_k) + (\beta_k-\beta) \tag{9.4}$$

and consider a point $x\in h(Q)$. If

$$E_{\beta,m}(x) < \nu_m(\beta) - 2(\beta_k-\beta)\ ,$$

by definition of e_k we have $e_k(x) = -\nabla E_{\beta_k,m}(x)$ and hence

$$\big(e_k(x), \nabla E_{\beta_k,m}(x)\big)_V \leq 0$$

for such x. If, on the other hand,

$$E_{\beta,m}(x) \geq \nu_m(\beta) - 2(\beta_k-\beta)\ ,$$

by assumption (9.4) estimate (9.2) is verified for x.

Observe that

$$\big(e_k(x), \nabla E_{\beta_k,m}(x)\big)_V \leq -\varphi_k(x)\big(\nabla E_{\beta,m}(x), \nabla E_{\beta_k,m}(x)\big)_V\ .$$

But, by (9.3), for points x satisfying (9.2) we can estimate

$$\begin{aligned}\big(\nabla E_{\beta,m}(x),&\nabla E_{\beta_k,m}(x)\big)_V \\ &= \|\nabla E_{\beta,m}(x)\|^2 - \big(\nabla E_{\beta,m}(x), \nabla G_{\beta_k,m}(x) - \nabla G_{\beta,m}(x)\big)_V \\ &\geq \frac{1}{2}\|\nabla E_{\beta,m}(x)\|^2 - \frac{1}{2}\|\nabla G_{\beta_k,m}(x) - \nabla G_{\beta,m}(x)\|^2 \\ &\geq \varepsilon - C\int_{\{t;|x(t)|\leq r\}} |\nabla \tilde{H}_{\beta_k,m}(x) - \nabla \tilde{H}_{\beta,m}(x)|^2 \, dt \\ &\geq \varepsilon - o(1) \ ,\end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$ by dominated convergence, uniformly in x. Hence, for large k, for any $x \in h(Q)$ we have

$$\frac{d}{dt} E_{\beta_k,m}\big(\Phi_k(x,t)\big)|_{t=0} = \big(e_k(x), \nabla E_{\beta_k,m}(x)\big)_V \leq 0 \ .$$

In particular, letting $h_t = \Phi_k(\cdot,t) \circ h$ for $t \geq 0$, estimate (9.4) holds for any $h_t, t \geq 0$. Moreover, $h_t \in \Gamma$ for any $t \geq 0$. Hence, by definition of $\nu_m(\beta)$, for any $t \geq 0$ we have

$$M(t) = \sup_{x \in Q} E_{\beta,m}\big(h_t(x)\big) \geq \nu_m(\beta) \ . \tag{9.5}$$

Together with (9.4), this implies that $M(t)$ is achieved only at points $x \in h_t(Q)$ satisfying (9.2). Moreover, $\varphi_k(x) = 1$ at such points and hence

$$\big(e_k(x), \nabla E_{\beta,m}(x)\big)_V = -\|E_{\beta,m}(x)\|^2 \leq -2\varepsilon$$

by (9.3).

Thus, $M(t)$ is strictly decreasing with

$$\frac{d}{dt} M(t) \leq -2\varepsilon < 0 \ ,$$

contradicting (9.5) for sufficiently large $t > 0$. This completes the proof. □

Proof of Theorem 9.1: For $\beta \in I_0$ with

$$\liminf_{m \to \infty} \frac{\partial}{\partial \beta} \nu_m(\beta) = C_\beta < \infty$$

let Λ and $(x_m)_{m \in \Lambda}$ be as in Lemma 9.4. For any m the function x_m solves (9.1) with $T(x_m) \leq C_\beta + 4$ and satisfies

$$|H(x_m(t)) - \beta| < \frac{\delta}{m} \ .$$

Hence the sequences $(x_m), (\dot{x}_m)$ are equi-bounded and equi-continuous;

By the theorem of Arzéla-Ascoli, therefore, a subsequence converges C^1-uniformly to a 1-periodic solution x of

$$\dot{x} = T\mathcal{J}\nabla H(x)$$

for some $T \geq 0$, with $H\big(x(t)\big) \equiv \beta$. Moreover, $A(x_m) \geq E_{\beta,m}(x_m) \geq \alpha$; hence $A(x) \geq \alpha > 0$. In particular, x is non-constant, $T > 0$, and $y(t) = x(t/T)$ is a T-periodic solution of (8.5) on S_β. □

9.5 Notes. (1°) Recently, Ginzburg [1] and Herman [1] have given an example of a smooth Hamiltonian H, possessing a smooth, compact energy level surface carrying no periodic orbit. Theorem 9.1 therefore is best possible. A related result is due to Kuperberg [1].
(2°) Further applications of the above method are given in Struwe [16], [17], Ambrosetti-Struwe [2]. However, an abstract statement would be quite cumbersome and in each instance, features that are particular to the given problem come into play. Common to all the above applications is a family of functionals $E_\beta \in C^1(V)$ depending monotonically on $\beta \in \mathbb{R}$. A key technical point is that a pseudo-gradient flow for E_β should be a pseudo-gradient flow also for E_{β_k} near points $x \in V$ such that $|E_{\beta_k}(x) - E_\beta(x)| \leq C|\beta_k - \beta|$, when $|\beta_k - \beta| \ll 1$.

10. Critical Points of Mountain Pass Type

Different critical points of functionals E sometimes can be distinguished by the topological type of their neighborhoods in the sub-level sets of E. This is the original idea of M. Morse which led to the development of what is now called Morse theory; see the Introduction. Hofer [2] has observed that such information about the topological type in some cases is available already from the minimax characterization of the corresponding critical value.

10.1 Definition. *Let $E \in C^1(V)$, and suppose u is a critical point of E with $E(u) = \beta$. We say u is of mountain pass type if for any neighborhood N of u the set $N \cap E_\beta$ is disconnected.*

With this notion available we can strengthen the assertion of Theorem 6.1 as follows. For convenience we recall

6.1 Theorem. *Suppose $E \in C^1(V)$ satisfies (P.-S.). Suppose*
(1°) $E(0) = 0$.
(2°) $\exists \rho > 0,\ \alpha > 0 :\ \|u\| = \rho \Rightarrow E(u) \geq \alpha$.
(3°) $\exists u_1 \in V :\ \|u_1\| \geq \rho$ and $E(u_1) < \alpha$.
Define

$$\Gamma = \Big\{ p \in C^0\left([0,1]; V\right)\ ;\ p(0) = 0,\ p(1) = u_1 \Big\}\ .$$

Then

$$\beta = \inf_{p \in \Gamma}\ \sup_{u \in p} E(u)$$

is a critical value.

Now we assert:

10.2 Theorem. *Under the hypotheses of Theorem 6.1 the following holds:*
(1°) either E admits a relative minimizer $u \neq 0$ with $E(u) = \beta$, or
(2°) E admits a critical point u of mountain pass type with $E(u) = \beta$.

Remark. Simple examples on $\mathbb{R}$ show that in general also case (1°) occurs; see Figure 10.1.

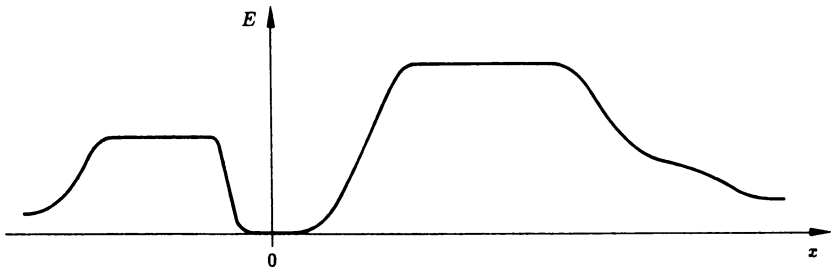

Fig. 10.1. A function possessing no critical point of mountain pass type

Another variant of Theorem 6.1 is related to results of Chang [4] and Pucci-Serrin [1], [2].

10.3 Theorem. *Suppose $E \in C^1(V)$ satisfies (P.-S.). Suppose 0 is a relative minimizer of E with $E(0) = 0$, and suppose that E admits a second relative minimizer $u_1 \neq 0$. Let Γ and β be defined as in Theorem 6.1. Then,*
(1°) either there exists a critical point $u \in K_\beta$ which is not of minimum type, or
(2°) the origin and u_1 can be connected in any neighborhood of the set of relative minimizers u of E with $E(u) = 0$. Necessarily then, $\beta = E(u_1) = E(0) = 0$.

The proofs of these results are quite similar in spirit.

Proof of Theorem 10.3. Let Γ and β be defined as in Theorem 6.1. Suppose that K_β consists entirely of relative minimizers of E. Then for each $u \in K_\beta$ there exists a neighborhood $N(u)$ such that

$$E(u) = \beta \leq E(v) \qquad \text{for all } v \in N(u) \ .$$

Let $N_0 = \bigcup_{u \in K_\beta} N(u)$, and for any neighborhood $\tilde{N}$ of K_β let $\varepsilon > 0$ and Φ be determined according to Theorem 3.4 for $\overline{\varepsilon} = 1$ and $N = N_0 \cap \tilde{N}$. Choosing $p \in \Gamma$ such that $p([0,1]) \subset E_{\beta+\varepsilon}$, by (1°),(3°) of Theorem 3.4 the path $p' = \Phi(\cdot, 1) \circ p \in \Gamma$ satisfies

$$p'([0,1]) \subset \Phi(E_{\beta+\varepsilon}, 1) \subset N \cup E_{\beta-\varepsilon} \subset N_0 \cup E_{\beta-\varepsilon} \ .$$

But N_0 and $E_{\beta-\varepsilon}$ by construction are disjoint, hence disconnected. Thus either $p'([0,1]) \subset N$ or $p'([0,1]) \subset E_{\beta-\varepsilon}$. Since the latter contradicts the definition of β we conclude that $p'([0,1]) \subset N \subset \tilde{N}$, whence $p(0) = 0$ and $p(1) = u_1$ can be connected in any neighborhood $\tilde{N}$ of K_β, as claimed. In particular, $0 \in K_\beta$, $u_1 \in K_\beta$, whence $\beta = E(0) = 0$ and also $E(u_1) = 0$. □

Proof of Theorem 10.2. Suppose, by contradiction, that K_β contains no relative minimizers nor critical points of mountain pass type. Then any $u \in K_\beta$ possesses a neighborhood $N(u)$ such that $N(u) \cap E_\beta$ is non-empty and (path-) connected. Moreover, $K_\beta \subset \overline{E_\beta}$. Now we have the following topological lemma; see Hofer [3; Lemma 1]:

10.4 Lemma. *Let (M,d) be a metric space and let K and Λ be non-empty subsets of M such that K is compact, Λ is open, and $K \subset \overline{\Lambda}$, the closure of Λ. Suppose $\{N(u)\ ;\ u \in K\}$ is an open cover of K such that $u \in N(u)$ and $N(u) \cap \Lambda$ is connected for each $u \in K$.*
Then there exists a finite, disjoint open cover $\{U_1, \dots, U_L\}$ of K such that $U_l \cap \Lambda$ for each l is contained in a connected component of Λ.

Proof. Choose $\delta > 0$ such that for any $u \in K$ we have

$$B_\delta(u) \subset N(\overline{u}) \qquad \text{for some } \overline{u} \in K\ . \tag{10.1}$$

For instance, we may choose a finite subcover $\{N(\overline{u}_i)\ ;\ 1 \le i \le I\}$ of $\{N(u)\ ;\ u \in K\}$ and let

$$\delta = \min_{u \in K} \max_{1 \le i \le I} \big\{ \operatorname{dist}(u, M \setminus N(\overline{u}_i)) \big\} > 0\ .$$

Define an equivalence relation $\overset{*}{\sim}$ on K by letting

$$u \overset{*}{\sim} \overline{u} \ \Leftrightarrow\ \begin{array}{l} \text{There exist a number } m \in \mathbb{N} \text{ and points } u_i \in K,\ 0 \le i \le \\ m+1, \text{ such that } u_0 = u,\ u_{m+1} = \overline{u} \text{ and } d(u_i, u_{i+1}) < \delta \text{ for} \\ 0 \le i \le m. \end{array}$$

Since K is compact there are only finitely many equivalence classes, say $K_1, \dots, K_L$. Let

$$U_l = \big\{ x \in M\ ;\ \operatorname{dist}(x, K_l) < \frac{\delta}{4} \big\}, \qquad 1 \le l \le L\ .$$

Then it is immediate that $U_k \cap U_l = \emptyset$ if $k \ne l$ and $\bigcup_{l=1}^{L} U_l \supset K$. It remains to show that each $U_l^* = U_l \cap \Lambda$ is contained in a connected component of Λ.

Define another equivalence relation $\sim$ on Λ by letting

$$v \sim w \quad \Leftrightarrow \quad \begin{array}{l} v \text{ and } w \text{ belong to the same connected} \\ \text{component of } \Lambda. \end{array}$$

Fix l and let $v, w \in U_l \cap \Lambda$. We wish to show that $v \sim w$. By definition of U_l and K_l there exists a finite chain $u_i \in K_l,\ 0 \le i \le m+1$, such that

$$d(v, u_0) < \frac{\delta}{4}, \quad d(w, u_{m+1}) < \frac{\delta}{4} ,$$

and

$$d(u_i, u_{i+1}) < \delta \ , \text{ for } 0 \le i \le m \ .$$

Set $\varepsilon = \delta - \max_{1 \le i \le m} d(u_i, u_{i+1}) > 0$. Since $K \subset \overline{\Lambda}$, for each $0 \le i \le m+1$ there exists $v_i \in \Lambda$ such that $d(u_i, v_i) < \frac{\varepsilon}{2} < \delta$, whence also

$$d(v_i, v_{i+1}) < \delta \qquad \text{for } 0 \le i \le m \ .$$

But now by (10.1) we have

$$v, v_0 \in \big(B_\delta(u_0) \cap \Lambda\big) \subset \big(N(\overline{u}_0) \cap \Lambda\big)$$

for some $\overline{u}_0 \in K$, and our assumption about the cover $\{N(u)\ ;\ u \in K\}$ implies that $v \sim v_0$. Similarly, $w \sim v_{m+1}$. Finally, for $0 \le i \le m$ we have

$$d(u_i, v_{i+1}) \le d(u_i, u_{i+1}) + d(u_{i+1}, v_{i+1}) < \delta \ ,$$

whence

$$v_i, v_{i+1} \in \big(B_\delta(u_i) \cap \Lambda\big) \subset \big(N(\overline{u}_i) \cap \Lambda\big) \ ,$$

and $v_i \sim v_{i+1}$ for all $i = 0, \ldots, m$. In conclusion

$$v \sim v_0 \sim v_1 \sim \ldots \sim v_{m+1} \sim w \ ,$$

and the proof is complete □

Proof of Theorem 10.2 (completed). Let $\{U_1, \ldots, U_L\}$ be a disjoint open cover of K_β as in Lemma 10.4 and set $N = \bigcup_{l=1}^{L} U_l$. Choose $\overline{\varepsilon} = \alpha > 0$ and let $\varepsilon > 0$, Φ be determined according to Theorem 3.4, corresponding to β, $\overline{\varepsilon}$, and N. Let $p \in \Gamma$ satisfy $p([0,1]) \subset E_{\beta+\varepsilon}$. Then $p' = \Phi(\cdot, 1) \circ p \in \Gamma$ and $p'([0,1]) \subset \Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N = E_{\beta-\varepsilon} \cup U_1 \cup \ldots \cup U_L$. Smoothing p' if necessary, we may assume $p' \in C^1$. Choose $\gamma \in]\beta-\varepsilon, \beta[$ such that γ is a regular value of $E \circ p'$.

Let $0 < t_1 < t_2 < \ldots < t_{2k-1} < t_{2k} < 1$ denote the successive pre-images of γ under $E \circ p'$, and let $I_j = [t_{2j-1}, t_{2j}]$, $1 \le j \le k$. Note that $p'|_{I_j} \subset \bigcup_{l=1}^{L} U_l$. Since the latter is a union of disjoint sets, for any j there is $l \in \{1, \ldots, L\}$ such that $p'(I_j) \subset U_l$. But $E \circ p'(\partial I_j) = \gamma < \beta$ and $U_l \cap E_\beta$ is connected. Hence we may replace $p'|_{I_j}$ by a path $\tilde{p}\colon I_j \to U_l \cap E_\beta$ with endpoints $\tilde{p}|_{\partial I_j} = p|_{\partial I_j}$, for any $j = 1, \ldots k$. In this way we obtain a path $\tilde{p} \in \Gamma$ such that $\sup_{u \in \tilde{p}} E(u) < \beta$, which yields the desired contradiction. □

Multiple Solutions of Coercive Elliptic Problems

We apply these results to a semilinear elliptic problem. Let Ω be a bounded domain in $\mathbb{R}^n$, $g\colon \mathbb{R} \to \mathbb{R}$ a continuous function. For $\lambda \in \mathbb{R}$ consider the problem

$$-\Delta u = \lambda u - g(u) \qquad \text{in } \Omega\ , \tag{10.2}$$

$$u = 0 \qquad \text{on } \partial\Omega\ , \tag{10.3}$$

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$ denote the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition. Then we may assert

10.5 Theorem. *Suppose g is locally Lipschitz with $g(0) = 0$ and assume that the map $u \mapsto \frac{g(u)}{|u|}$ is non-decreasing with*
(1°) $\lim_{u\to 0} \frac{g(u)}{u} = 0$, *and*
(2°) $\lim_{|u|\to\infty} \frac{g(u)}{u} = \infty$.
Then for any $\lambda > \lambda_2$ problem (10.2), (10.3) admits at least three distinct non-trivial solutions.

Note that by assumption (1°) the problem (10.2), (10.3) for any $\lambda \in \mathbb{R}$ admits $u \equiv 0$ as (trivial) solution. Moreover, by (2°) , the functional E related to (10.2), (10.3) is coercive. The latter stands in contrast with the examples studied earlier in this chapter. As a consequence, the existence of multiple solutions for problem (10.2), (10.3) heavily depends on the behavior of the functional E near $u = 0$, governed by the parameter λ, whereas in previous examples the behavior near ∞ had been responsible for the nice multiplicity results obtained.

Theorem 10.5 is due to Ambrosetti-Mancini [1] and Struwe [5]. Later, Ambrosetti-Lupo [1] were able to simplify the argument significantly, and we shall basically follow their approach in the proof below. See also Rabinowitz [11; Theorem 2.42], Chang [1; Theorem 3], and Hofer [1] for related results.

If g is odd, then for $\lambda_k < \lambda \leq \lambda_{k+1}$ problem (10.2), (10.3) possesses at least k pairs of distinct non-trivial solutions, see for example Ambrosetti [1]. (This can also be deduced from a variant of Theorem 5.7 above, applied to the sub-level set $M = E_0$ of the functional E related to (10.2), (10.3). Note that M is forwardly invariant under the pseudo-gradient flow for E and hence the trajectories of this flow are complete in forward time. Moreover, for $\lambda_k < \lambda \leq \lambda_{k+1}$ by (1°) it follows that the genus $\hat{\gamma}(M) \geq k$.)

Without any symmetry assumption optimal multiplicity results for (10.2), (10.3) are not known. However, results of Dancer [2] suggest that in general even for large λ one can expect no more than four non-trivial solutions.

Proof of Theorem 10.5. Let $u_\pm = \min\{u > 0\ ;\ \frac{g(\pm u)}{\pm u} \geq \lambda\}$. Then we may replace g by the truncated function

$$\hat{g}(u) = \begin{cases} g(u), & -u_- \leq u \leq u_+ \\ \lambda u, & u < -u_- \text{ or } u_+ < u. \end{cases}$$

Observe that if u satisfies

$$-\Delta u = \lambda u - \hat{g}(u) = \begin{cases} \lambda u - g(u), & \text{if } u \in [-u_-, u_+] \\ 0, & \text{else} \end{cases}$$

then by the weak maximum principle in fact u satisfies (10.2), (10.3).

Thus we may assume that $|g(x,u) - \lambda u| \le c$ and $\big(\lambda u - g(u)\big)u \ge 0$ for all $u \in \mathbb{R}$. Hence the functional

$$E(u) = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 - \lambda |u|^2\big)\, dx + \int_\Omega G(x,u)\, dx$$

is well-defined, and $E \in C^1\big(H_0^{1,2}(\Omega)\big)$. Moreover, (P.-S.) is satisfied. In fact, E is coercive, and, if we identify $H_0^{1,2}(\Omega)$ with its dual via the inner product, DE is of the form id + compact. Therefore, (P.-S.) follows from Proposition 2.2.

In a first step we now want to exhibit a positive solution $\bar{u}$ of (10.2), (10.3), and a negative solution $\underline{u}$, respectively. This can be done in various ways by using the methods outlined in the previous chapters. For instance, we might truncate the nonlinearity $\lambda u - g(u)$ below or above 0 and proceed as in the proof of Theorem 6.2. However, we can also use the trivial solution $u = 0$ as a sub-(super-) solution to problem (10.2), (10.3) and minimize the functional E in the cone of non-negative (non-positive) functions, as we did earlier in Sections I.2.3, I.2.4. We choose this latter approach.

Let $\overline{u}, \underline{u}$ minimize E in $\overline{M} = \{u \in H_0^{1,2}(\Omega)\ ;\ u \ge 0\}$, respectively $\underline{M} = \{u \in H_0^{1,2}(\Omega)\ ;\ u \le 0\}$. Then, since 0 is a trivial solution to (10.2), (10.3), by Theorem I.2.4 the functions $\underline{u}, \overline{u}$ also solve (10.2), (10.3). In particular, $\underline{u}, \overline{u} \in C^{2,\alpha}(\Omega)$ for some $\alpha > 0$; see Appendix B. Moreover, if $\lambda > \lambda_1$ and if $\varphi_1 > 0$, $\|\varphi_1\|_{L^2} = 1$, denotes a normalized eigenfunction corresponding to λ_1, we have

$$E(\varepsilon\varphi_1) = \frac{1}{2}(\lambda_1 - \lambda)\varepsilon^2 - o(\varepsilon^2)\ ,$$

where $o(s)/s \to 0$ as $s \to 0$, and this is < 0 for sufficiently small $|\varepsilon| \ne 0$; hence $\underline{u}, \overline{u} \not\equiv 0$, and in fact, by the strong maximum principle (Theorem B.4 of Appendix B), the functions $\underline{u}$, $\overline{u}$ cannot have an interior zero.

We claim: $\underline{u}$ and $\overline{u}$ are relative minimizers of E in $H_0^{1,2}(\Omega)$. It suffices to show that for sufficiently small $\rho > 0$ we have

$$E(\overline{u}) \le E(u) \text{ for all } u \in \overline{M}_\rho := \{u \in H_0^{1,2}(\Omega)\ ;\ \|(\overline{u} - u)_-\|_{H_0^{1,2}} \le \rho\}\ ,$$

where $(s)_\pm = \pm\max\{\pm s, 0\}$. Fix $\rho > 0$ and let $u_\rho \in \overline{M}_\rho$ be a minimizer of E in $\overline{M_\rho}$. Then

$$\langle v, DE(u_\rho)\rangle \ge 0$$

for all $v \ge 0$; that is, u_ρ is a (weak) supersolution to (10.2), (10.3), satisfying

$$-\Delta u_\rho \ge \lambda u_\rho - g(u_\rho)\ .$$

Choosing $v = -(u_\rho)_- \geq 0$ as testing function, we obtain

$$\int_\Omega |\nabla u_{\rho_-}|^2 - \lambda |u_{\rho_-}|^2 \, dx + \int_\Omega g(u_{\rho_-}) u_{\rho_-} \, dx \leq 0 \ .$$

But by (1°) there is a constant $c \geq 0$ such that $g(u)\, u \geq -c|u|^2$ for all u; thus

$$\int_\Omega |\nabla (u_{\rho_-})|^2 - (\lambda + c) |u_{\rho_-}|^2 \, dx \leq 0 \ .$$

Let $\Omega_{\rho_-} = \{x \in \Omega \ ; \ u_\rho < 0\}$. Then, since $\overline{u} > 0$, we have $\mathcal{L}^n(\Omega_{\rho_-}) \to 0$ as $\rho \to 0$. Now by Hölder's and Sobolev's inequalities we can estimate

$$\begin{aligned} \int_{\tilde{\Omega}} |u|^2 \, dx &\leq \big(\mathcal{L}^n(\tilde{\Omega})\big)^{\frac{2}{n}} \left(\int_{\tilde{\Omega}} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \\ &\leq C(n) \big(\mathcal{L}^n(\tilde{\Omega})\big)^{\frac{2}{n}} \int_{\tilde{\Omega}} |\nabla u|^2 \, dx \end{aligned}$$

for all $u \in H_0^{1,2}(\tilde{\Omega})$, and for any $\tilde{\Omega} \subset \Omega$. Thus,

$$\lambda_{\rho_-} := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} \ ; \ u \in H_0^{1,2}(\Omega_{\rho_-}) \setminus \{0\} \right\} \to \infty \text{ as } \rho \to 0 \ ,$$

and for $\rho > 0$ sufficiently small such that $\lambda_{\rho_-} > (\lambda + c)$ we obtain $u_{\rho_-} \equiv 0$; that is, $u_\rho \geq 0$. Hence $E(u_\rho) \geq E(\overline{u})$, and $\overline{u}$ is a relative minimizer of E in $\overline{M}_\rho$, whence also in $H_0^{1,2}(\Omega)$. The same conclusion is valid for $\underline{u}$.

Now let

$$\Gamma = \left\{ p \in C^0 \left([0,1]; H_0^{1,2}(\Omega) \right) \ ; \ p(0) = \underline{u}, p(1) = \overline{u} \right\}$$

and denote

$$\beta = \inf_{p \in \Gamma} \sup_{u \in \rho} E(u) \ .$$

If $\beta \neq 0$ we are done, because Theorem 10.3 guarantees either the existence of infinitely many relative minimizers or the existence of a critical point $u \in K_\beta$ not of minimum type and thus distinct from $\underline{u}$, $\overline{u}$. Since $\beta \neq 0$, this point u must also be distinct from the trivial solution $u = 0$.

If $\beta = 0$, Theorem 10.3 may yield the third critical point $u = 0$. However; by Theorem 10.2 there exists a critical point u with $E(u) = \beta$ of minimum or mountain pass type. But for $\lambda > \lambda_2$, by assumption (1°) and since the map $u \mapsto \frac{g(u)}{|u|}$ is non-decreasing, the set E_0 where E is negative is connected; thus, $u = 0$ is not of one of these types, and the proof is complete. □

Remark. Since in the case $\beta = 0$ we have to show only that $K_\beta \neq \{0\}$, instead of appealing to Theorem 10.2 it would suffice to show that, if $u = 0$ were the only critical point besides $\underline{u}$ and $\overline{u}$, then 0 can be avoided by a path joining $\underline{u}$ with $\overline{u}$ without increasing energy. For $\lambda > \lambda_2$ this is easily shown by a direct construction in the spirit of Theorem 10.2.

Notes 10.6. (1°) In the context of Hamiltonian systems (8.5), Ekeland-Hofer [1] have applied Theorem 10.2 to obtain the existence of periodic solutions with prescribed minimal period for certain convex Hamiltonian functions H; see also Girardi-Matzeu [1], [2]. Similar applications to semilinear wave equations as considered in Section I.6.6 have recently been given by Salvatore [1].
(2°) Generalizations of Theorem 10.2 to higher-dimensional minimax methods like Theorem 6.3 have been obtained by Bahri-Lions [1], Lazer-Solimini [1], and Viterbo [2]. See Remark 7.3 for an application of these results. Recently, Ghoussoub [1], [2] has presented a unified approach to results in the spirit of Theorems 10.2 and 10.3.

11. Non-differentiable Functionals

Sometimes a functional $E: V \to \mathbb{R} \cup \{\pm\infty\}$ may fail to be Fréchet-differentiable on V but may only be Gâteaux-differentiable on its domain

$$\mathrm{Dom}(E) = \{u \in V \ ; \ E(u) < \infty\}$$

in direction of a dense space of "testing functions" $T \subset V$.

11.1 Nonlinear scalar field equations: The zero mass case. As an example we consider the problem

$$-\Delta u = g(u) \qquad \text{in } \mathbb{R}^n, \ n \geq 3 \ , \tag{11.1}$$

with the asymptotic boundary condition

$$u(x) \to 0 \qquad (|x| \to \infty) \ . \tag{11.2}$$

The associated energy integral is

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx - \int_{\mathbb{R}^n} G(u) \, dx \ , \tag{11.3}$$

where, as usual, we denote

$$G(u) = \int_0^u g(v) \, dv$$

a primitive of g. In the case of "positive mass", that is

$$\limsup_{u \to 0} g(u)/u \leq -m < 0 \ ,$$

similar to the problem studied in Section I.4.1, problem (11.1), (11.2) can be dealt with as a variational problem in $H^{1,2}(\mathbb{R}^n)$, where E is differentiable; see Berestycki-Lions [1]. In contrast to Section I.4.1, however, now we do not exclude the "0-mass case"

$$\frac{g(u)}{u} \to 0 \qquad (u \to 0) \ .$$

Then a natural space on which to study E is the space $D^{1,2}(\mathbb{R}^n)$; that is, the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \ .$$

By Sobolev's inequality, an equivalent characterization is

$$D^{1,2}(\mathbb{R}^n) = \left\{ u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \ ; \ \nabla u \in L^2(\mathbb{R}^n) \right\} \ .$$

Note that, unless G satisfies the condition

$$|G(u)| \le c|u|^{\frac{2n}{n-2}} \ ,$$

the functional E may be infinite on a dense set of points in this space and hence cannot be Fréchet-differentiable on $D^{1,2}(\mathbb{R}^n)$. To overcome this difficulty, in Struwe [6], [7] a variant of the Palais-Smale condition was introduced and a critical point theory was developed, giving rise to existence results for saddle points for a broad class of functionals where standard methods fail. Below, we give an outline of the abstract scheme of this method; then we apply it to our model problem (11.1), (11.2) above to obtain a simple proof for the following result of Berestycki and Lions [1], [2].

11.2 Theorem. *Suppose g is continuous with primitive $G(u) = \int_0^u g(v)\, dv$ and satisfies the conditions*
(1°) $-\infty \le \limsup_{u\to 0} g(u)u/|u|^{\frac{2n}{n-2}} \le 0$,
(2°) $-\infty \le \limsup_{|u|\to\infty} g(u)u/|u|^{\frac{2n}{n-2}} \le 0$,
and suppose there exists a constant ξ_1 such that
(3°) $G(\xi_1) > 0$.
Moreover, assume that g is odd, that is,
(4°) $g(-u) = -g(u)$.
Then there exist infinitely many radially symmetric solutions $u_l \in D^{1,2}(\mathbb{R}^n)$ of (11.1), (11.2) and $E(u_l) \to \infty$ as $l \to \infty$.

Remark. Observe that by the maximum principle we may replace g by the function $\tilde{g}$, given by

$$\tilde{g}(u) = \begin{cases} g(\zeta^*), & \text{if } u > \zeta^* \\ g(u), & \text{if } |u| \le \zeta^* \\ g(-\zeta^*), & \text{if } u < -\zeta^* \end{cases}$$

where $\zeta^* = \inf\{\zeta \ge \zeta_1 \ ; \ g(\zeta) \le 0\} \le +\infty$. Indeed, if u solves (11.1), (11.2) for $\tilde{g}$, by the maximum principle u solves (11.1), (11.2) for g. Hence we may assume that instead of (2°) g satisfies the stronger hypothesis

(5°) $\lim_{|u|\to\infty} g(u)u/|u|^{\frac{2n}{n-2}} = 0$
together with the assumption
(6°) $\exists \zeta_2 > 0 : G(\zeta) > 0$ for all $\zeta, |\zeta| > \zeta_2$.

For the proof of Theorem 11.2 we now follow Struwe [6].

11.3 The abstract scheme. Suppose that

$$E\colon \mathrm{Dom}(E) \subset V \to \mathbb{R}$$

is a densely defined functional on a Banach space V with norm $\|\cdot\|$. Moreover, assume there is a family $(T_L)_{L\in\mathbb{N}}$ of Banach spaces

$$T_1 \subset \ldots \subset T_L \subset T_{L+1} \subset \ldots \subset V$$

with norms $\|\cdot\|_L$ such that

$$\|u\| \le \ldots \le \|u\|_{L+1} \le \|u\|_L \qquad \text{for } u \in T_L\ ,$$

and whose union is dense in V:

$$T := \bigcup_{L\in\mathbb{N}} T_L \overset{\text{dense}}{\subset} V\ . \tag{11.4}$$

(By default, all topological statements refer to the norm-topology of V.) Also suppose that for any $u \in \mathrm{Dom}(E)$ the restricted functional $E|_{\{u\}+T_L} \in C^1(T_L)$, for any $L \in \mathbb{N}$, and the partial derivative

$$D_L E\colon \mathrm{Dom}(E) \ni u \mapsto D_L E(u) \in T_L^*$$

is continuous in the topology of V for any $L \in \mathbb{N}$.

We define $u \in \mathrm{Dom}(E) \subset V$ to be critical if $D_L E(u) = 0$ for all $L \in \mathbb{N}$, and we denote

$$K_\beta = \{u \in \mathrm{Dom}(E)\ ;\ E(u) = \beta,\ D_L E(u) = 0,\ \forall L \in \mathbb{N}\}$$

the set of critical points with energy β.

Suppose that E satisfies the following variant of the Palais-Smale condition:

$(\widetilde{\text{P.-S.}})$ Any sequence (u_m) in $\mathrm{Dom}(E)$ such that $E(u_m) \to \beta$, while $D_L E(u_m) \to 0$ in T_L^* $(m \to \infty)$, for any $L \in \mathbb{N}$, possesses an accumulation point in K_β.

Note the following:

11.4 Lemma. *Suppose V satisfies (11.4) and $E\colon \mathrm{Dom}(E) \subset V \to \mathbb{R}$ satisfies ($\widetilde{\text{P.-S.}}$). Then for any $\beta \in \mathbb{R}$ the set K_β is compact and any neighborhood N of K_β contains a member of the family*

$$N_{\beta,L} = \{u \in \mathrm{Dom}(E)\ ;\ |E(u) - \beta| < 1/L,\ \|D_L E(u)\|_L^* < 1/L\},\ L > 0\ .$$

Moreover, the system

$$U_{\beta,\rho} = \{u \in V\ ;\ \exists v \in K_\beta : \|u - v\| < \rho\}$$

is a fundamental system of neighborhoods of K_β.

Proof. By ($\widetilde{\text{P.-S.}}$) any sequence (u_m) in K_β is relatively compact and accumulates at some point $u \in K_\beta$. To prove the second assertion, suppose by contradiction that for some neighborhood N of K_β and any $L \in \mathbb{N}$ there is a point $u_L \in N_{\beta,L} \setminus N$. Consider the sequence (u_L). Since for any L' we have

$$\|D_{L'} E(u_L)\|_{L'}^* \le \|D_L E(u_L)\|_L^* \le 1/L \to 0$$

as $L \to \infty,\ L \ge L'$, from ($\widetilde{\text{P.-S.}}$) we conclude that (u_L) accumulates at a point $u \in K_\beta$, contrary to assumption. The proof for $U_{\beta,\rho}$ is similar. □

Denote

$$\mathrm{Reg}_L(E) = \{u \in \mathrm{Dom}(E)\ ;\ D_L E(u) \neq 0\}$$

the set of regular points of E with respect to variations in T_L. Then exactly as in Lemma 3.2, using continuity of the partial derivative $D_L E$, we can construct a locally Lipschitz continuous pseudo-gradient vector field $v_L\colon \mathrm{Reg}_L(E) \to T_L$ for E satisfying the conditions

$$\|v_L(u)\| < 2\min\{1, \|D_L E(u)\|_L^*\}\ ,$$
$$\langle v_L(u), D_L E(u)\rangle > \min\{1, \|D_L E(u)\|_L^*\}\|D_L E(u)\|_L^*\ .$$

Now the deformation lemma Theorem 3.4 can be carried over easily. Given $\beta \in \mathbb{R}$, $N \supset K_\beta$ we determine $L \in \mathbb{N}$, $\rho > 0$ such that

$$N \supset U_{\beta,\rho} \supset U_{\beta,\rho/2} \supset N_{\beta,L}\ .$$

Choose a locally Lipschitz continuous function $\eta,\ 0 \le \eta \le 1$, such that $\eta = 0$ if $\|D_L E(u)\|_L^* \le \frac{1}{2L}$ and such that $\eta(u) = 1$ if $\|D_L E(u)\|_L^* \ge \frac{1}{L}$. Thus, in particular, we have $\eta(u) = 1$ for $u \notin N_{\beta,L}$ satisfying $|E(u) - \beta| \le \frac{1}{L}$. Let Φ_L be the flow corresponding to the vector field $e_L(u) = -\eta(u) v_L(u)$, defined by solving the initial value problem

$$\frac{\partial}{\partial t}\Phi_L(u,t) = e_L\big(\Phi_L(u,t)\big)$$
$$\Phi_L(u,0) = u$$

for $u \in \mathrm{Dom}(E)$, $t \geq 0$. By local Lipschitz continuity and uniform boundedness of e_L, the flow Φ_L exists globally on $\mathrm{Dom}(E) \times [0, \infty[$, is continuous, and fixes critical points of E. Moreover, E is non-increasing along flow-lines, and we have

$$\Phi_L(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N \ ,$$

respectively

$$\Phi_L(E_{\beta+\varepsilon} \setminus N, 1) \subset E_{\beta-\varepsilon} \ ,$$

where $\varepsilon = \min\left\{\frac{1}{L}, \frac{\rho}{4L^2}\right\}$; see the proof of Theorem 3.4.

Note that we do not require E to be continuous on its domain. Thus, we have to be careful with truncating the vector field e_L outside some energy range. However, with this crude tool already, many of our abstract existence results may be carried over.

Proof of Theorem 11.2. Let us now implement the above scheme with E given by (11.3) on

$$V = \{u \in D^{1,2}(\mathbb{R}^n) \ ; \ u(x) = u(|x|)\} =: D^{1,2}_{rad}(\mathbb{R}^n) \ ,$$

with norm $\|\cdot\| = \|\cdot\|_{D^{1,2}}$. (We focus on radially symmetric solutions to remove translation invariance.) Also let

$$T_L = \left\{u \in D^{1,2}_{rad}(\mathbb{R}^n) \ ; \ u(x) = 0 \ \text{ for } |x| \geq L\right\} \ ,$$

with norm $\|\cdot\|_L = \|\cdot\| = \|\cdot\|_{D^{1,2}}$, $L \in \mathbb{N}$. Note that in this way $T = \bigcup_{L\in\mathbb{N}} T_L$ simply consists of all functions $u \in D^{1,2}_{rad}(\mathbb{R}^n)$ with compact support. Since variations in T_L for any L only involve the evaluation of g, respectively G on a compact domain $B_L(0)$, it is clear that $E(u+\cdot)$ is Fréchet-differentiable in T_L, for any $u \in \mathrm{Dom}(E)$, any $L \in \mathbb{N}$. Moreover, the differential $D_L E\colon \mathrm{Dom}(E) \to T_L^*$ is continuous in the topology of V for any $L \in \mathbb{N}$.

Note that by radial symmetry any function $u \in D^{1,2}_{rad}(\mathbb{R}^n)$ is represented by a function (indiscriminately denoted by u), which is continuous on $\mathbb{R}^n \setminus \{0\}$. Moreover, by Hölders's inequality there holds

$$\begin{aligned} |u(x)|^2 &\leq \int_{|x|}^{\infty} \frac{d}{dr}|u(r)|^2 \, dr \\ &\leq 2\left(\int_{|x|}^{\infty} r^{(2-n)n-1} \, dr\right)^{\frac{1}{n}} \left(\int_{|x|}^{\infty} |u(r)|^{\frac{2n}{n-2}} r^{n-1} \, dr\right)^{\frac{n-2}{2n}} \\ &\qquad \times \left(\int_{|x|}^{\infty} |\frac{d}{dr}u(r)|^2 r^{n-1} \, dr\right)^{\frac{1}{2}} \\ &\leq C \, |x|^{(2-n)} \|u\|^2 \ , \end{aligned} \tag{11.6}$$

for any $u \in D^{1,2}_{rad}(\mathbb{R}^n)$.

Decompose $g = g_+ - g_-$, $G = G_+ - G_-$ with $g_\pm(u)u = \max\{\pm g(u)u, 0\}$ and $G_\pm(u) = \int_0^u g_\pm(v)\,dv$. We assert that, if $u_m \rightharpoonup u$ weakly in $D^{1,2}_{rad}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} G_+(u)\,dx = \lim_{m\to\infty} \int_{\mathbb{R}^n} G_+(u_m)\,dx\ . \tag{11.7}$$

Indeed, by (11.6), for any $\delta > 0$ there exists $R > 0$ such that $|u_m(x)| \le \delta$ for $|x| \ge R$. Moreover, by assumption (1°) , for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$G_+(u_m) \le \varepsilon |u_m|^{\frac{2n}{n-2}}\ ,$$

if $|u| \le \delta$. Hence for large R we can estimate

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_R(0)} G_+(u_m)\,dx &\le \varepsilon \int_{\mathbb{R}^n\setminus B_R(0)} |u_m|^{\frac{2n}{n-2}}\,dx \\ &\le C\,\varepsilon\,\|u_m\|^{\frac{2n}{n-2}} \le C\,\varepsilon\ ,\end{aligned}$$

uniformly in m.

On the other hand, by assumption (2°) , for any $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that for all $u \in \mathbb{R}$ there holds

$$G_+(u) \le \varepsilon\,|u|^{\frac{2n}{n-2}} + C(\varepsilon)\ .$$

Hence for $\Omega \subset \mathbb{R}^n$ with sufficiently small measure $\mathcal{L}^n(\Omega) < \frac{\varepsilon}{C(\varepsilon)}$, we have

$$\int_\Omega G_+(u_m)\,dx \le \varepsilon \int_\Omega |u|^{\frac{2n}{n-2}}\,dx + C(\varepsilon)\mathcal{L}^n(\Omega) \le C\,\varepsilon\ ,$$

uniformly in m, and (11.7) follows by Vitali's convergence theorem.

Since $G_- \ge 0$, by Fatou's lemma of course also

$$\int_{\mathbb{R}^n} G_-(u)\,dx \le \liminf_{m\to\infty} \int_{\mathbb{R}^n} G_-(u_m)\,dx\ ,$$

and together with (11.7) we obtain that, if $u_m \rightharpoonup u$ weakly, then

$$\int_{\mathbb{R}^n} G(u)\,dx \ge \limsup_{m\to\infty} \int_{\mathbb{R}^n} G(u_m)\,dx\ . \tag{11.8}$$

Moreover, in order to verify ($\widetilde{\text{P.-S.}}$) we need the following estimate reminiscent of the "Pohožaev identity"; see Lemma III.1.4.

11.5 Lemma. *Suppose that (1°) and (2°) of Theorem 11.2 hold. Then any weak solution $u \in D^{1,2}_{rad}(\mathbb{R}^n)$ of equation (11.1) satisfies the estimate*

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \frac{2n}{n-2} \int_{\mathbb{R}^n} G(u) \, dx \geq -\infty \ .$$

Proof. By (2°) and our local regularity result Lemma B.3 of Appendix B, any weak solution to (11.2) is in L^p_{loc} and hence also in $H^{2,p}$ locally, for any $p < \infty$. Moreover, by (1°),(2°) we have $G_+(u) \in L^1(\mathbb{R}^n)$. Testing Equation (11.1) with the function $x \cdot \nabla u$ (the generator of the family $u_R(x) = u(Rx)$ of dilations of the function u) we may write the product as

$$\begin{aligned} \frac{n-2}{2}|\nabla u|^2 - \operatorname{div}\left(x\frac{|\nabla u|^2}{2} - \nabla u \, (x \cdot \nabla u)\right) &= (x \cdot \nabla u) \, \Delta u \\ &= -(x \cdot \nabla u) \, g(u) = -x \cdot \nabla G(u) = -\operatorname{div}\big(xG(u)\big) + nG(u) \ . \end{aligned} \tag{11.9}$$

Integrating over $B_R(0)$ and using the radial symmetry of u, we thus obtain that

$$\begin{aligned} \int_{B_R(0)} |\nabla u|^2 \, dx &= \frac{2n}{n-2} \int_{B_R(0)} G(u) \, dx \\ &\quad - \frac{2R}{n-2} \int_{\partial B_R(0)} \big(\frac{1}{2}|\nabla u|^2 + G(u)\big) \, do \\ &\geq \frac{2n}{n-2} \int_{B_R(0)} G(u) \, dx \\ &\quad - \frac{2R}{n-2} \int_{\partial B_R(0)} \big(\frac{1}{2}|\nabla u|^2 + G_+(u)\big) \, do \ . \end{aligned}$$

Since $\nabla u \in L^2(\mathbb{R}^n)$, $G_+(u) \in L^1(\mathbb{R}^n)$, if we let $R \to \infty$ in a suitable way the boundary integral tends to zero. Moreover,

$$\begin{aligned} \int_{B_R(0)} |\nabla u|^2 \, dx &\to \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \ , \\ \int_{B_R(0)} G_+(u) \, dx &\to \int_{\mathbb{R}^n} G_+(u) \, dx \ , \end{aligned}$$

while from Beppo Levi's theorem it follows that

$$\int_{\mathbb{R}^n} G_-(u) \, dx \geq \limsup_{R \to \infty} \int_{B_R(0)} G_-(u) \, dx \ .$$

The proof is complete. □

Remark that under scaling $u \mapsto u_R(x) = u(Rx)$ the functional E behaves like

$$E(u_R) = \frac{1}{2} R^{2-n} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx - R^{-n} \int_{\mathbb{R}^n} G(u) \, dx \ ;$$

that is,

$$\frac{d}{dR}E(u_R)|_{R=1} = \frac{2-n}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx + n\int_{\mathbb{R}^n} G(u)\,dx\ .$$

Hence we may perform a preliminary normalization of admissible functions by restricting E to the set

$$M = \{u \in \mathrm{Dom}(E)\ ;\ u \neq 0,\ \int_{\mathbb{R}^n} |\nabla u|^2\,dx = \frac{2n}{n-2}\int_{\mathbb{R}^n} G(u)\,dx\}$$

of functions which are stationary for E with respect to dilations. Note that for $u \in M$ we have

$$E(u) = \frac{1}{n}\int_{\mathbb{R}^n} |\nabla u|^2\,dx\ ;$$

that is, $E|_M$ is continuous and coercive with respect to the norm in $D^{1,2}(\mathbb{R}^n)$.

Moreover, $E|_M$ satisfies the following compactness condition:

11.6 Lemma. *Suppose that for a sequence (u_m) in M as $m \to \infty$ we have $E(u_m) \to \beta$ while $D_L E(u_m) \to 0 \in T_L^*$ for any L. Then (u_m) accumulates at a critical point $u \in M$ of E and $E(u) = \beta$. That is, E satisfies ($\widetilde{\text{P.-S.}}$) on M (while it seems unlikely that one can even show boundedness of a ($\widetilde{\text{P.-S.}}$) sequence in general).*

Proof. Let (u_m) be a ($\widetilde{\text{P.-S.}}$) sequence for E in M. By coerciveness, (u_m) is bounded and we may assume that $u_m \rightharpoonup u$ weakly in $D^{1,2}_{rad}(\mathbb{R}^n)$, which implies strong convergence $g(u_m) \to g(u)$ in $L^1(\Omega)$ for any $\Omega \subset\subset \mathbb{R}^n$. Thus for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ and sufficiently large L, as $m \to \infty$ we have

$$\begin{aligned}\langle \varphi, D_L E(u_m)\rangle &= \int_{\mathbb{R}^n} \big(\nabla u_m \nabla\varphi - g(u_m)\varphi\big)\,dx \\ &\to \int_{\mathbb{R}^n} \big(\nabla u \nabla\varphi - g(u)\varphi\big)\,dx = \langle \varphi, D_L E(u)\rangle = 0\ ;\end{aligned}$$

that is, $u \in D^{1,2}_{rad}(\mathbb{R}^n)$ is a critical point of E and hence weakly solves (11.1). By Lemma 11.5 and (11.8) therefore

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u|^2\,dx &\geq \frac{2n}{n-2}\int_{\mathbb{R}^n} G(u)\,dx \geq \frac{2n}{n-2}\limsup_{m\to\infty}\int_{\mathbb{R}^n} G(u_m)\,dx \\ &= \limsup_{m\to\infty}\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx \geq \int_{\mathbb{R}^n} |\nabla u|^2\,dx\ .\end{aligned}$$

Here we have also used the normalization condition for $u_m \in M$. Hence $u_m \to u$ strongly in $D^{1,2}_{rad}(\mathbb{R}^n)$, and also $\int_{\mathbb{R}^n} G(u_m)\,dx \to \int_{\mathbb{R}^n} G(u)\,dx$; in particular, $u \in M$ and $E(u_m) \to E(u) = \beta$, as claimed. □

Now we investigate the set M more closely.

Denote $k\colon \mathbb{R}_+ \times \mathrm{Dom}(E) \to \mathbb{R}$ the mapping

$$
\begin{aligned}
k(R,u) &= \frac{2R^{n+1}}{2-n}\,\frac{d}{d\rho}E(u_\rho)|_{\rho=R} \\
&= R^2\int_{\mathbb{R}^n}|\nabla u|^2\,dx - \frac{2n}{n-2}\int_{\mathbb{R}^n}G(u)\,dx\ .
\end{aligned}
\tag{11.10}
$$

Then for any $L \in \mathbb{N}$ and any $u \in \mathrm{Dom}(E)$ we have $k(\,\cdot\,, u+\,\cdot\,) \in C^1(\mathbb{R}_+ \times T_L)$. Moreover, by Hölder's inequality the partial derivatives $\partial_L k(R,u) \in T_L^*$, $\frac{\partial}{\partial R}k(R,u)$, and $\frac{\partial^2}{\partial R^2}k(R,u)$ are continuous and uniformly bounded on bounded sets $\{(R,u) \in \mathbb{R}_+ \times M + T_L\ ;\ R + \|u\| \le C\}$. Finally, for $u \in M$, at $R=1$ we have

$$
\frac{\partial}{\partial R}k(R,u)|_{R=1} = 2\int_{\mathbb{R}^n}|\nabla u|^2\,dx = 2n\,E(u) > 0\ .
$$

Thus, by the implicit function theorem, for any $\beta > 0$ and any $L \in \mathbb{N}$ there exists $\rho = \rho_{\beta,L} > 0$ and a continuous map $R = R_{\beta,L}$ on a neighborhood

$$
V_{\beta,L} = \{u \in M\ ;\ \beta/2 \le E(u) \le 2\beta\} + B_{2\rho}(0;T_L)
$$

such that $R(v+\cdot) \in C^1(T_L)$ and $k\big(R(v),v\big) \equiv 0$ for $v \in V_{\beta,L}$; that is, $v_{R(v)}(x) = v\big(R(v)x\big) \in M$. Denote

$$
\pi_{\beta,L}\colon V_{\beta,L} \to M
$$

the map $v \mapsto \pi_{\beta,L}(v) = v_{R(v)}$. Remark that $\pi_{\beta,L}$ is continuous. For the next lemma let

$$
\begin{aligned}
K_\beta &= \{u \in M\ ;\ E(u) = \beta,\ D_L E(u) = 0 \text{ for all } L\}\ , \\
N_{\beta,L} &= \{u \in M\ ;\ |E(u)-\beta| < 1/L,\ \|D_L E(u)\|_L^* < 1/L\}\ .
\end{aligned}
$$

Note that by Lemma 11.6 the assertions of Lemma 11.4 hold for K_β, $N_{\beta,L}$ as above. We now construct a pseudo-gradient flow for E on M.

11.7 Lemma. *For any $\beta > 0$, any $\overline{\varepsilon} > 0$ and any neighborhood N of K_β there exist $\varepsilon \in]0,\overline{\varepsilon}[$ and a continuous family $\Phi\colon M \times [0,1] \to M$ of odd continuous maps $\Phi(\,\cdot\,,t)\colon M \to M$ such that*
(1°) $\Phi(u,t) = u$ if $D_L E(u) = 0$ for all $L \in \mathbb{N}$, or $t = 0$, or if $|E(u)-\beta| \ge \overline{\varepsilon}$,
(2°) $E\big(\Phi(u,t)\big)$ is non-increasing in t for any $u \in M$,
(3°) $\Phi(E_{\beta+\varepsilon} \setminus N, 1) \subset E_{\beta-\varepsilon}$.

Proof. Choose integers $L' < L$ such that

$$
N \supset N_{\beta,L'} \supset N_{\beta,L}
$$

and let $v_L\colon \{u \in M\ ;\ D_L E(u) \ne 0\} \to T_L$ be an odd, continuous pseudo-gradient vector field for E, satisfying

$$\|v_L(u)\|_L < 2\min\{1, \|D_L E(u)\|_L^*\} ,$$
$$\langle v_L(u), D_L E(u)\rangle > \min\{1, \|D_L E(u)\|_L^*\}\|D_L E(u)\|_L^* ,$$

for all $u \in M$ such that $D_L E(u) \neq 0$. Let η, φ be continuous cut-off functions $0 \leq \eta, \varphi \leq 1$, $\eta(u) = \eta(-u)$, $\eta \equiv 0$ on $N_{\beta,L}$, $\eta \equiv 1$ off $N_{\beta,L'}$, $\varphi(s) = 0$ for $|s-\beta| \geq 2\varepsilon$, $\varphi(s) = 1$ for $|s-\beta| < \varepsilon$, where $\varepsilon \leq \min\{\frac{\bar{\varepsilon}}{2}, \frac{1}{4L}\}$ will be determined in the sequel. We truncate v_L as usual and let

$$e_L(u) = -\varphi\big(E(u)\big)\eta(u)v_L(u) .$$

Note that $e_L: M \to T_L$ is odd and continuous. Let $\rho_{\beta,L}$, and $\pi_{\beta,L}$ be defined as above. Provided that $\varepsilon \leq \frac{\beta}{4}$, for $t \leq \rho_{\beta,L}$ then we may let

$$\Phi(u,t) = \pi_{\beta,L}\big(u + te_L(u)\big) .$$

Φ is continuous, odd, and satisfies (1°). Moreover, for fixed $u \in M$ the term $E\big(\Phi(u,t)\big)$ is differentiable in t. Indeed, letting

$$R(t) = R\big(u + te_L(u)\big),\ u_R(x) = u\big(Rx\big)$$

for brevity, we have

$$\begin{aligned}\frac{d}{dt}E\big(\Phi(u,t)\big)\big|_{t=t_0} &= \frac{d}{dt}E\big((u+te_L(u))_{R(t)}\big)|_{t=t_0}\\ &= \left(\frac{d}{dR}E\big(\big(u+t_0e_L(u)\big)_R\big)\big|_{R=R(t_0)}\right)\frac{d}{dt}R(t)|_{t=t_0}\\ &\quad + \big\langle\big(e_L(u)\big)_{R(t_0)}, D_L E\big(\Phi(u,t_0)\big)\big\rangle .\end{aligned}$$

Since $\big(u+t_0e_L(u)\big)_{R=R(t_0)} = \Phi(u,t_0) \in M$ the first term vanishes. Moreover, the second up to a factor $-\varphi\big(E(u)\big)\eta(u)$ equals

$$\begin{aligned}&\big\langle\big(v_L(u)\big)_{R(t_0)}, D_L E\big(\Phi(u,t_0)\big)\big\rangle\\ &\quad = R(t_0)^{2-n}\int_{\mathbb{R}^n}\nabla\big(u+t_0e_L(u)\big)\nabla v_L(u)\,dx\\ &\qquad - R(t_0)^{-n}\int_{\mathbb{R}^n} g\big(u+t_0e_L(u)\big)v_L(u)\,dx\\ &\quad =: \big\langle v_L(u), D_L E(u)\big\rangle - \gamma(t_0) ,\end{aligned}$$

where the last line defines the "error function" γ.

Note that $t \to R(t)$ is differentiable with $\frac{d}{dt}R(t)$ uniformly bounded on bounded sets and that by condition (5°) of the theorem and Vitali's convergence theorem also

$$\int_{B_L(0)}\big|g\big(u+te_L(u)\big) - g(u)\big|^{\frac{2n}{n+2}}\,dx \to 0 \text{ as } t \to 0 ,$$

uniformly on bounded sets of functions $u \in M$. Hence also the error

$$\gamma(t) \le c\big(|R(t)-1|+|t|\big) + \left|\int_{\mathbb{R}^n} \big(g\big(u+te_L(u)\big)-g(u)\big)e_L(u)\,dx\right|$$
$$\to 0 \text{ as } t \to 0\ .$$

In particular, we can achieve that uniformly in $u \in E_{\beta+\overline{\varepsilon}}$ we have $\gamma(t) \le \frac{1}{2L^2}$ for $0 \le t \le \overline{t}$ sufficiently small. By choice of φ and η this implies that

$$\frac{d}{dt}E\big(\Phi(u,t)\big) \le -\frac{\varphi\big(E(u)\big)\eta(u)}{2L^2} \qquad \text{for } u \in M,\ 0 \le t \le \overline{t}\ .$$

Hence, with $\varepsilon \le \frac{\overline{t}}{4L^2}$, (2°) and (3°) follow. Rescaling time we may assume $\overline{t}=1$. □

Finally, we can conclude the proof of Theorem 11.2. For $l \in \mathbb{N}$ let

$$\Sigma_l = \{A \subset M\ ;\ A \text{ closed},\ A=-A,\ \gamma(A) \ge l\}\ ,$$

where γ denotes the Krasnoselskii genus introduced in Section 5.1, and define

$$\beta_l = \inf_{A\in\Sigma_l}\ \sup_{u\in A} E(u)\ .$$

11.8 Lemma. *(1°) For any $l \in \mathbb{N}$ the class Σ_l is nonvoid, in particular, $M \ne \emptyset$. (2°) The numbers β_l are critical values of E for any $l \in \mathbb{N}$. Moreover, $\beta_l \to \infty$ as $l \to \infty$.*

Proof. (1°) Fix $l \in \mathbb{N}$. By condition (6°) on G we can find an l-dimensional subspace $W \subset C^\infty_{0,rad}(\mathbb{R}^n)$ and a constant $\alpha_1 > 0$ such that for $w \in S = \{w \in W\ ;\ \|w\| = \alpha_1\}$ we have

$$\int_{\mathbb{R}^n} G(w)\,dx > 0\ .$$

By (11.10) we can find $\tau > 0$ such that

$$k(\tau, w) < 0 \tag{11.11}$$

for all $w \in S$. Scaling x with τ, if necessary, we may assume that $\tau = 1$. Since $W \subset C^\infty_0$ there exists another constant α_2 such that $\|w\|_{L^\infty} \le \alpha_2$ for $w \in S$. Consider the truncation mapping $\delta\colon W \to D^{1,2}_{rad}(\mathbb{R}^n)$ given by

$$\delta(w)(x) = \begin{cases} \alpha_2, & w(x) > \alpha_2, \\ w(x), & |w(x)| \le \alpha_2, \\ -\alpha_2, & w(x) < -\alpha_2. \end{cases}$$

Note that δ is continuous and odd. Since the functions $\delta(w)$ are uniformly bounded and have uniform compact support, clearly

$$\int_{\mathbb{R}^n} G\big(\delta(w)\big)\,dx \le c \tag{11.12}$$

uniformly in $w \in W$. On the other hand, for any $w \in S$ there holds

$$\int_{\mathbb{R}^n} |\nabla \delta(\mu w)|^2 \, dx \to \infty \qquad (\mu \to \infty) \,, \tag{11.13}$$

as is easily verified.

For $w \in W$, with $\|w\| \geq \alpha_1$, let

$$J(w) = k\big(1, \delta(w)\big)$$

and extend J continuously to an even map $J: W \to \mathbb{R}$ such that $J(w) < 0$ for $\|w\| < \alpha_1$. (Note that by (11.11) we have $J(w) = k(1, w) < 0$ for $w \in S$; that is, for $\|w\| = \alpha_1$.) By (11.12), (11.13) the set $\Omega = \{w \in W \,;\; J(w) < 0\}$ then is an open, bounded, symmetric neighborhood of $0 \in W$. Hence, from Proposition 5.2 we deduce that the boundary A of Ω relative to W has genus $\gamma(A) \geq l$. Since the mapping δ is odd and continuous, by supervariance of the genus, Proposition 5.4(4°), also $\gamma\big(\delta(A)\big) \geq l$. Moreover, since $J(A) = \{0\}$ and $\delta(A) \not\ni 0$, we clearly have $\delta(A) \subset M$, concluding the proof of (1°).

(2°) By part (1°) and Lemma 11.7 the numbers β_l are well-defined and critical; see the proof of Theorem 4.2. To show that $\beta_l \to \infty$ $(l \to \infty)$ assume by contradiction that $\beta_l \leq \beta$ uniformly in l. Thus, we can find a sequence of sets $A_l \in \Sigma_l$ such that $E(u) \leq 2\beta$ for $u \in A_l$, $l \in \mathbb{N}$. Letting $A = \bigcup_l A_l$ by Proposition 5.4(2°) and Proposition 5.3 there exists an infinite sequence of mutually orthogonal vectors $u_m \in A$. By coerciveness and uniform boundedness of E on A, there holds $\|u_m\| \leq C$ for all m, and hence we may extract a weakly convergent subsequence (u_m) (relabeled). By mutual orthogonality, $u_m \rightharpoonup 0$ weakly $(m \to \infty)$. Decomposing $g = g_+ - g_-$ as above, with $u \cdot g_\pm(u) = \max\{0, \pm u \cdot g(u)\}$, by (11.7) therefore

$$\int_{\mathbb{R}^n} G_+(u_m) \, dx \to 0 \qquad (m \to \infty) \,. \tag{11.14}$$

But for any $u \in M$

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx &\leq \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \frac{2n}{n-2} \int_{\mathbb{R}^n} G_-(u) \, dx \\ &= \frac{2n}{n-2} \int_{\mathbb{R}^n} G_+(u) \, dx \,. \end{aligned} \tag{11.15}$$

Thus, $u_m \to 0$ strongly in $D^{1,2}_{rad}(\mathbb{R}^n)$ as $m \to \infty$. On the other hand, by assumptions (1°), (2°) there exists a constant $c > 0$ such that for all u there holds

$$g(u)u \leq c|u|^{\frac{2n}{n-2}} \,,$$

and consequently

$$G(u) \leq c|u|^{\frac{2n}{n-2}} \,.$$

Hence, for $u \in M$, by the Sobolev embedding theorem there holds

$$\|u\|^2 = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \frac{2n}{n-2} \int_{\mathbb{R}^n} G(u) \, dx \leq c\|u\|^{\frac{2n}{n-2}}_{L^{\frac{2n}{n-2}}} \leq c\|u\|^{\frac{2n}{n-2}} \, .$$

Dividing by $\|u\|^2$, we conclude that

$$\|u\| \geq c > 0$$

is uniformly bounded away from 0 for $u \in M$, and a contradiction to (11.14), (11.15) results. This concludes the proof. □

11.9 Notes. In a recent paper, Duc [1] has developed a variational approach to singular elliptic boundary value problems which is similar to the method outlined above. Duc only requires continuity of directional derivatives of E. However, in exchange, only a weaker form of the deformation lemma can be established; see Duc [1; Lemma 2.5]. Very interesting new ideas in this regard can also be found in Duc [2].

Lack of differentiability is encountered in a different way when dealing with functionals involving a combination of a differentiable and a convex term, as in the case of free boundary problems. For such functionals E, using the notion of sub-differential introduced in Section I.6, a differential DE may be defined as a set-valued map. Suitable extensions of minimax techniques to such problems have been obtained by Chang [2] and Szulkin [1]. More generally, using the concept of generalized gradients introduced by Clarke [1], [5], Chang [2] develops a complete variational theory also for Lipschitz maps satisfying a (P.-S.)-type compactness condition. In Ambrosetti-Struwe [2] these results, in combination with the technique of parameter variations described in Section II.9 above, are used to establish the existence of steady vortex rings in an ideal fluid for a prescribed, positive, non-decreasing vorticity function. Previously, this problem had been studied by Fraenkel-Berger [1] by a constrained minimization technique; however, their approach gave rise to a Lagrange multiplier that could not be controlled.

12. Lusternik-Schnirelman Theory on Convex Sets

In applications we also frequently encounter functionals on closed and convex subsets of Banach spaces. In fact, this is the natural setting for variational inequalities where the class of admissible functions is restricted by inequality constraints; see Sections I.2.3, I.2.4. Functionals on closed convex sets also arise in certain geometric problems, as we have seen in our discussion of the classical Plateau problem in Sections I.2.7–I.2.10. In fact it was precisely for the latter problem, with the aim of re-deriving the mountain pass lemma for minimal surfaces due to Morse-Tompkins [1], [2] and Shiffman [2], [3], that variational methods for functionals on closed convex sets were first systematically developed; see Struwe [9], [18].

Suppose M is a closed, convex subset of a Banach space V, and suppose that $E\colon M \to \mathbb{R}$ possesses an extension $E \in C^1(V;\mathbb{R})$ to V. For $u \in M$ define

$$g(u) = \sup_{\substack{v\in M\\ \|u-v\|<1}} \langle u - v, DE(u)\rangle$$

as a measure for the slope of E in M. Clearly, if $M = V$ we have $g(u) = \|DE(u)\|$. More generally, we obtain

12.1 Lemma. *If $E \in C^1(V)$, the function g is continuous in M.*

Proof. Suppose $u_m \to u$ $(m \to \infty)$, where $u_m, u \in M$. Then for any $v \in M$ such that $\|u - v\| < 1$ and sufficiently large m there also holds $\|u_m - v\| < 1$. Hence for any such $v \in M$ we may estimate

$$\begin{aligned}\langle u - v, DE(u)\rangle &= \lim_{m\to\infty} \langle u_m - v, DE(u_m)\rangle \\ &\le \limsup_{m\to\infty} g(u_m)\ .\end{aligned}$$

Passing to the supremum with respect to v in this inequality, we infer that

$$g(u) \le \limsup_{m\to\infty} g(u_m)\ .$$

On the other hand, if for $\varepsilon_m \searrow 0$ we choose $v_m \in M$ such that $\|u_m - v_m\| < 1$ and

$$\langle u_m - v_m, DE(u_m)\rangle \ge g(u_m) - \varepsilon_m\ ,$$

by convexity of M, also the vectors

$$\begin{aligned}w_m &= \|u_m - u\|u_m + \big(1 - \|u_m - u\|\big)v_m \\ &= u_m + \big(1 - \|u_m - u\|\big)(v_m - u_m)\end{aligned}$$

belong to M and satisfy

$$\begin{aligned}\|w_m - u\| &\le \|u_m - u\| + \big(1 - \|u_m - u\|\big)\|v_m - u_m\| \\ &< \|u_m - u\| + \big(1 - \|u_m - u\|\big) = 1\ ,\end{aligned}$$

while

$$\|v_m - w_m\| = \|u_m - u\|\,\|v_m - u_m\| \le \|u_m - u\| \to 0\ .$$

Thus,

$$\begin{aligned}g(u) &\ge \limsup_{m\to\infty} \langle u - w_m, DE(u)\rangle \\ &= \limsup_{m\to\infty} \langle u_m - v_m, DE(u_m)\rangle = \limsup_{m\to\infty} g(u_m)\ ,\end{aligned}$$

and the proof is complete. □

12.2 Definition. *A point $u \in M$ is critical if $g(u) = 0$, otherwise u is regular. If $E(u) = \beta$ for some critical point $u \in M$ of E, the value β is critical; otherwise β is regular.*

This definition coincides with the definition of regular or critical points (values) of a functional given earlier, if $M = V$.

Moreover, as usual we let

$$\begin{aligned} M_\beta &= \{u \in M \ ; \ E(u) < \beta\} \ , \\ K_\beta &= \{u \in M \ ; \ E(u) = \beta, \ g(u) = 0\} \ , \\ N_{\beta,\delta} &= \{u \in M \ ; \ |E(u) - \beta| < \delta, \ g(u) < \delta\} \ , \\ U_{\beta,\rho} &= \{u \in M \ ; \ \exists v \in K_\beta : \|u - v\| < \rho\} \end{aligned}$$

denote the sub-level sets, critical sets and families of neighborhoods of K_β, for any $\beta \in \mathbb{R}$.

12.3 Definition. *E satisfies the Palais-Smale condition on M if the following is true:*

$(P.\text{-}S.)_M$ *Any sequence (u_m) in M such that $|E(u_m)| \le c$ uniformly, while $g(u_m) \to 0$ $(m \to \infty)$, is relatively compact.*

12.4 Lemma. *Suppose E satisfies $(P.\text{-}S.)_M$. Then for any $\beta \in \mathbb{R}$ the set K_β is compact. Moreover, the families $\{N_{\beta,\delta} \ ; \ \delta > 0\}$, respectively $\{U_{\beta,\rho} \ ; \ \rho > 0\}$ constitute fundamental systems of neighborhoods of K_β.*

The proof is identical with that of Lemma 2.3.

Denote

$$\tilde{M} = \{u \in M \ ; \ g(u) \neq 0\}$$

the set of regular points of E, and let

$$K = \{u \in M \ ; \ g(u) = 0\} = M \setminus \tilde{M}$$

be the set of critical points .

12.5 Definition. *A locally Lipschitz vector field $v \colon \tilde{M} \to V$ is a pseudo-gradient vector field for E on M if there exists $c > 0$ such that*
(1°) $u + v(u) \in M$,
(2°) $\|v(u)\| < \min\{1, g(u)\}$,
(3°) $\langle v(u), DE(u)\rangle < -c \min\{1, g(u)\} g(u)$,
for all $u \in \tilde{M}$.

Arguing as in the proof of Lemma 3.2 we establish:

and since M is convex, this initial value problem may be solved by Euler's method.

Assertions (1°), (2°) are trivially satisfied by definition of e, (3°) is proved exactly as in Theorem 3.4. □

The results from the preceding sections now may be carried over to functionals on closed, convex sets. In particular, similar to Theorem 10.3 we have:

12.8 Theorem. *Suppose M is a closed, convex subset of a Banach space V, $E \in C^1(V)$ satisfies (P.-S.)$_M$ on M, and suppose E admits two distinct relative minima u_1, u_2 in M. Then either $E(u_1) = E(u_2) = \beta$ and u_1, u_2 can be connected in any neighborhood of the set of relative minima $u \in M$ of E with $E(u) = \beta$, or there exists a critical point $\overline{u}$ of E in M which is not a relative minimizer of E.*

Applications to Semilinear Elliptic Boundary Value Problems

Here we will not enter into a detailed discussion of applications of these methods to the Plateau problem for minimal surfaces and for surfaces of constant mean curvature for which they were developed. The reader will find this material in the lecture notes of Struwe [18], devoted exclusively to this topic, and in Chang-Eells [1], Jost-Struwe [1], Struwe [13].

Nor will we touch upon applications to variational inequalities. In this context, variational methods first seem to have been applied by Miersemann [1] to eigenvalue problems in a cone; see also Kučera [1] and Quittner [1]. Using the methods outlined above, a unified approach to equations and inequalities can be achieved.

Instead, we re-derive Amann's [2], [3] famous "three solution theorem" on the existence of "unstable" solutions of semilinear elliptic boundary value problems, confined in an order interval between sub- and supersolutions, in the variational case.

12.9 Theorem. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ and $g\colon \mathbb{R} \to \mathbb{R}$ is of class C^1, satisfying the growth condition*

$$(1^\circ) \qquad |g_u(u)| \le c\big(1 + |u|^{p-2}\big), \qquad \text{for some } \ p < \frac{2n}{n-2} \ .$$

Also suppose that the problem

$$(2^\circ) \qquad -\Delta u = g(u) \qquad \text{in } \Omega \ ,$$

$$(3^\circ) \qquad u = 0 \qquad \text{on } \partial\Omega \ ,$$

admits two pairs of sub- and supersolutions $\underline{u}_1 \le \overline{u}_1 \le \underline{u}_2 \le \overline{u}_2 \in C^2 \cap H_0^{1,2}(\Omega)$. *Then either* $\overline{u}_1$ *or* $\underline{u}_2$ *weakly solves (2°), (3°), or (2°), (3°) admits at least three distinct solutions* $u_i,\ \underline{u}_1 \le u_i \le \overline{u}_2,\ i = 1, 2, 3$.

Proof. By Theorem I.2.4 the functional $E \in C^1\big(H_0^{1,2}(\Omega)\big)$ related to (2°), (3°), given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega G(u)\, dx\ ,$$

admits critical points u_i which are relative minima of E in the order intervals $\underline{u}_i \le u_i \le \overline{u}_i,\ i = 1, 2$. Let $\underline{u} = \underline{u}_1,\ \overline{u} = \overline{u}_2$, and define

$$M = \{u \in H_0^{1,2}(\Omega)\ ;\ \underline{u} \le u \le \overline{u}\ \text{ a.e.}\}\ .$$

Observe that – unless $\overline{u}_1$ respectively $\underline{u}_2$ solves (2°),(3°) – u_1 and u_2 are also relative minima of E in M. To see this, for $i = 1, 2$ consider

$$M_i = \{u \in H_0^{1,2}(\Omega)\ ;\ \underline{u}_i \le u \le \overline{u}_i \text{ almost everywhere}\}$$

and for $\rho > 0$, as in the proof of Theorem 10.5, let

$$M_i^\rho = \{u \in M\ ;\ \exists v \in M_i : \|u - v\|_{H_0^{1,2}} \le \rho\}\ .$$

Note that M_i^ρ is closed and convex, hence weakly closed, and E is coercive and weakly lower semi-continuous on M_i^ρ with respect to $H_0^{1,2}(\Omega)$. By Theorem I.1.2, therefore, E admits relative minimizers $u_i^\rho \in M_i^\rho, i = 1, 2$, for any $\rho > 0$, and there holds

$$\big\langle (u_1^\rho - \overline{u}_1)_+, DE(u_1^\rho)\big\rangle \le 0 \le \big\langle (\underline{u}_2 - u_2^\rho)_+, DE(u_2^\rho)\big\rangle\ ,$$

with $(s)_+ = \max\{0, s\}$, as usual. Subtracting the relations

$$\big\langle (u_1^\rho - \overline{u}_1)_+, DE(\overline{u}_1)\big\rangle \ge 0 \ge \big\langle (\underline{u}_2 - u_2^\rho)_+, DE(\underline{u}_2)\big\rangle\ ,$$

we obtain that

$$\begin{aligned}
\|(u_1^\rho - \overline{u}_1)_+\|_{H_0^{1,2}}^2 &= \int_\Omega \big|\nabla(u_1^\rho - \overline{u}_1)_+\big|^2\, dx \le \int_\Omega \big(\big(g(u_1^\rho) - g(\overline{u}_1)\big)(u_1^\rho - \overline{u}_1)_+\, dx \\
&\le \int_\Omega \left(\int_0^1 g_u(\overline{u}_1 + s(u_1^\rho - \overline{u}_1))\, ds\right)(u_1^\rho - \overline{u}_1)_+^2\, dx \\
&\le \sup_{v \in M_1^\rho} \|g_u(v)\|_{L^{\frac{p}{p-2}}} \|(u_1^\rho - \overline{u}_1)_+\|_{L^p}^2 \\
&\le C\big(1 + \sup_{v \in M_1^\rho} \|v\|_{L^p}^{p-2}\big)\cdot \\
&\qquad \mathcal{L}^n\big(\{x\ ;\ u_1^\rho(x) > \overline{u}_1(x)\}\big)^{2\gamma}\ \cdot \|(u_1^\rho - \overline{u}_1)_+\|_{L^{\frac{2n}{n-2}}}^2
\end{aligned}$$

where $\gamma = 1 - \frac{n-2}{2n}p > 0$, and an analogous estimate for $(\underline{u}_2 - u_2^\rho)_+$. By Sobolev's embedding theorem

12.6 Lemma. *There exists a pseudo-gradient vector field $v\colon \tilde{M} \to V$, satisfying (3°) of Definition 12.5 with $c = \frac{1}{2}$.*

Proof. For $u \in \tilde{M}$ choose $w = w(u) \in M$ such that

$$\|u - w\| < \min\{1, g(u)\} \ , \tag{12.1}$$

$$\langle u - w, DE(u)\rangle > \frac{1}{2}\min\{1, g(u)\}g(u) \ . \tag{12.2}$$

Now, as in the proof of Lemma 3.2, let $\{U_\iota \ ; \ \iota \in I\}$ be a locally finite open cover of $\tilde{M}$ such that for any $\iota \in I$ and any $u \in U_\iota$ the conditions (12.1), (12.2) hold with $w = w(u_\iota)$, for some $u_\iota \in U_\iota$. Then let $\{\varphi_\iota \ ; \ \iota \in I\}$ be a locally Lipschitz partition of unity on $\tilde{M}$ subordinate to $\{U_\iota\}$ and define

$$v(u) = \sum_{\iota \in I} \varphi_\iota(u)\big(w(u_\iota) - u\big) \ ,$$

for $u \in V$. The resulting v is a pseudo-gradient vector field on $\tilde{M}$, as claimed. □

12.7 Theorem (Deformation Lemma). *Suppose $M \subset V$ is closed and convex, $E \in C^1(V)$ satisfies (P.-S.)$_{\mathrm{M}}$ on M, and let $\beta \in \mathbb{R}$, $\overline{\varepsilon} > 0$ be given. Then for any neighborhood N of K_β there exist $\varepsilon \in]0, \overline{\varepsilon}[$ and a continuous deformation $\Phi\colon M \times [0,1] \to M$ such that*
(1°) $\Phi(u,t) = u$ if $g(u) = 0$, or if $t = 0$, or if $|E(u) - \beta| \geq \overline{\varepsilon}$;
(2°) $E\big(\Phi(u,t)\big)$ is non-increasing in t, for any $u \in M$;
(3°) $\Phi(M_{\beta+\varepsilon}, 1) \subset M_{\beta-\varepsilon} \cup N$, respectively $\Phi(M_{\beta+\varepsilon} \setminus N, 1) \subset M_{\beta-\varepsilon}$.

Proof. For $\varepsilon < \min\{\overline{\varepsilon}/2, \delta/4\}$, where

$$N \supset U_{\beta,\rho} \supset U_{\beta,\rho/2} \supset N_{\beta,\delta} \ ,$$

as in the proof of Theorem 3.4, let η be locally Lipschitz with $0 \leq \eta \leq 1$, $\eta(u) = 0$ in $N_{\beta,\delta/2}$, $\eta(u) = 1$ on $N_{\beta,\delta}$ and choose a smooth cut-off function $0 \leq \varphi \leq 1$ such that $\varphi(s) = 1$ for $|s - \beta| \leq \varepsilon$, $\varphi(s) = 0$ for $|s - \beta| \geq 2\varepsilon$. Define

$$e(u) = \begin{cases} \eta(u)\varphi\big(E(u)\big)v(u), & u \in \tilde{M} \\ 0, & u \in K \ . \end{cases}$$

The vector field e is Lipschitz continuous, uniformly bounded, and induces a global flow $\Phi\colon M \times [0,1] \to M$ such that

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(u,t) &= e\big(\Phi(u,t)\big) \\ \Phi(u,0) &= 0 \ . \end{aligned}$$

Note that since v (and therefore e) satisfies the condition

$$u + v(u) \in M \ , \text{ for all } u \in \tilde{M} \ ,$$

$$\|v\|_{L^p} \le c\|v\|_{L^{\frac{2n}{n-2}}} \le c \inf_{w\in M_1} \big(\|v-w\|_{L^{\frac{2n}{n-2}}} + \|w\|_{L^{\frac{2n}{n-2}}}\big)$$
$$\le c \inf_{w\in M_1} \|v-w\|_{H_0^{1,2}} + C \le C < \infty$$

for $v \in M_1^\rho, \rho \le 1$. Similarly, $\|(u_1^\rho - \overline{u}_1)_+\|_{L^{\frac{2n}{n-2}}} \le c\|(u_1^\rho - \overline{u}_1)_+\|_{H^{1,2}}$, whence with a uniform constant C for all $\rho > 0$ there holds

$$\|(u_1^\rho - \overline{u}_1)_+\|^2_{H_0^{1,2}} \le C\mathcal{L}^n\big(\{x\ ;\ u_1^\rho(x) > \overline{u}_1(x)\}\big)^{2\gamma}\|(u_1^\rho - \overline{u}_1)_+\|^2_{H_0^{1,2}}\ , \tag{12.5}$$

and an analogous estimate for $(\underline{u}_2 - u_2^\rho)_+$.

As $\rho \to 0$ the functions u_i^ρ accumulate at minimizers $\tilde{u}_i$ of E in $M_i, i = 1, 2$. Arguing as in the proof of Theorem I.2.4, and using the regularity result Lemma B.3 of Appendix B, these functions $\tilde{u}_i$ are classical solutions of (2°),(3°). Now, if $\overline{u}_1, \underline{u}_2$ do not solve problem (2°),(3°), then in particular we have $\tilde{u}_1 \ne \overline{u}_1, \tilde{u}_2 \ne \underline{u}_2$. From the strong maximum principle we then infer that $\tilde{u}_1 < \overline{u}_1, \underline{u}_2 < \tilde{u}_2$. Hence

$$\mathcal{L}^n\big(\{x; u_1^\rho(x) > \overline{u}_1(x)\}\big) \to 0 \qquad (\rho \to 0)\ ,$$

and from (12.5) it follows that $u_1^\rho \in M_1$ for sufficiently small $\rho > 0$, showing that u_1 is relatively minimal for E in M. Similarly for u_2. Thus, u_1 and u_2 are relative minima of E in M.

By Theorem 12.8 the functional E either admits infinitely many relative minima in M or possesses at least one critical point $u_3 \in M$ which is not a relative minimizer of E, hence distinct from $u_1,\ u_2$.

Finally, observe that any critical point u of E in M weakly solves (2°), (3°): Indeed for any $\varphi \in H_0^{1,2}(\Omega),\ \varepsilon > 0$, let

$$v_\varepsilon = \min\big\{\overline{u}, \max\{\underline{u}, u + \varepsilon\varphi\}\big\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in M$$

with $\varphi^\varepsilon = \max\{0, u + \varepsilon\varphi - \overline{u}\} \ge 0$, $\varphi_\varepsilon = \max\{0, \underline{u} - (u + \varepsilon\varphi)\} \ge 0$, as in the proof of Theorem I.2.4. Then, if $u \in M$ is critical, as $\varepsilon \to 0$ we have

$$\langle u - v_\varepsilon, DE(u)\rangle \le \|u - v_\varepsilon\|_{H_0^{1,2}} \sup_{\substack{v\in M \\ \|u-v\|_{H_0^{1,2}} \le 1}} \langle u - v, DE(u)\rangle$$
$$\le O(\varepsilon)g(u) = 0\ .$$

From this inequality the desired conclusion follows exactly as in the proof of Theorem I.2.4. □

12.10 Notes. A different variational approach to this result was suggested by Chang [5], who used the regularizing properties of the heat flow to reduce the problem to a variational problem on an open subset of a suitable Banach space to which the standard methods could be applied.

A combination of topological degree and variational methods was used by Hofer [1] to obtain even higher multiplicity results for certain problems.

Chapter III

Limit Cases of the Palais-Smale Condition

Condition (P.-S.) may seem rather restrictive. Actually, as Hildebrandt [4; p. 324] records, for quite a while many mathematicians felt convinced that inspite of its success in dealing with one-dimensional variational problems like geodesics (see Birkhoff's Theorem I.4.4, for example, or Palais' [3] work on closed geodesics), the Palais-Smale condition could never play a role in the solution of "interesting" variational problems in higher dimensions.

Recent advances in the Calculus of Variations have changed this view and it has become apparent that the methods of Palais and Smale apply to many problems of physical and/or geometric interest and – in particular – that the Palais-Smale condition will in general hold true for such problems in a broad range of energies. Moreover, the failure of (P.-S.) at certain energy levels reflects highly interesting phenomena related to internal symmetries of the systems under study. Geometrically, we might attribute the non-compactness of certain Palais-Smale sequences to the "separation of spheres", or to "bubbling", perhaps associated with a "change in topology". With some twist of imagination, and speaking again in physical terms, we might observe "phase transitions" or "particle creation" at the energy levels where (P.-S.) fails.

Such phenomena seem to have first been observed by Sacks-Uhlenbeck [1] and – independently – by Wente [5] in the context of two-dimensional harmonic maps, respectively, in the context of surfaces of prescribed constant mean curvature. (See Sections 5 and 6 below.) In these cases the terms "separation of spheres" and "bubbling" have a clear geometric meaning. More recently, Sedlacek [1] has uncovered similar results also for Yang-Mills connections. If interpreted appropriately, very early indications of such phenomena already may be found in the work of Douglas [2], Morse-Tompkins [2] and Shiffman [2] on minimal surfaces of higher genus and/or connectivity. In this case, a "change in topology" in fact sometimes may be observed even physically as one tries to realize a multiply connected or higher genus minimal surface in a soap film experiment. See Jost-Struwe [1] for a modern approach to these results.

Mathematically, it seems that non-compact group actions give rise to these effects. In physics and geometry, of course, such group actions naturally arise as "symmetries" from the requirements of scale or gauge invariance; in particular, in the examples of Sacks-Uhlenbeck and Wente cited above, from conformal invariance.

A symmetry may either be "manifest" or "broken" by interaction terms. As we shall see, existence results in the spirit of Theorem II.6.1 for problems with non-compact internal symmetries often depend on the extent to which the symmetry is broken or perturbed, as measured by whether or not certain mountain-pass energy levels differ from the energy levels where "bubbling" may occur. As in the case of the non-compact minimization problems studied in Section I.4, in many instances the latter question can be answered by comparing the problem at hand with a suitable (family of) "limiting problem(s)" where the symmetry is acting. We start with a simple example.

1. Pohožaev's Non-existence Result

Let Ω be a domain in $\mathbb{R}^n$, $n > 2$. Consider the limit case $p = 2^* = \frac{2n}{n-2}$ in Theorem I.2.1. Given $\lambda \in \mathbb{R}$, we would like to solve the problem

$$-\Delta u = \lambda u + u|u|^{2^*-2} \qquad \text{in } \Omega \ , \tag{1.1}$$

$$u > 0 \qquad \text{in } \Omega \ , \tag{1.2}$$

$$u = 0 \qquad \text{on } \partial\Omega \ . \tag{1.3}$$

(Note that in order to be consistent with the literature, in this section we reverse the sign of λ as compared with Section I.2.1 or Section II.5.8.) As in Theorem I.2.1 we can approach this problem by a direct method and attempt to obtain non-trivial solutions of (1.1), (1.3) as relative minima of the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda|u|^2\right)\, dx \ ,$$

on the unit sphere in $L^{2^*}(\Omega)$,

$$M = \{u \in H_0^{1,2}(\Omega) \ ; \ \|u\|_{L^{2^*}}^{2^*} = 1\} \ .$$

Equivalently, we may seek to minimize the Sobolev quotient

$$S_\lambda(u;\Omega) = \frac{\int_\Omega \left(|\nabla u|^2 - \lambda|u|^2\right)\, dx}{\left(\int_\Omega |u|^{2^*}\, dx\right)^{2/2^*}}, \ u \in H_0^{1,2}(\Omega) \setminus \{0\}.$$

Note that for $\lambda = 0$, as in Section I.4.4,

$$S(\Omega) = \inf_{\substack{u \in H_0^{1,2}(\Omega) \\ u \neq 0}} S_0(u;\Omega) = \inf_{\substack{u \in H_0^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega |u|^{2^*}\, dx\right)^{2/2^*}}$$

is related to the (best) Lipschitz constant for the Sobolev embedding $H_0^{1,2}(\Omega) \to L^{2^*}(\Omega)$.

Recall that for any $u \in H_0^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$ the ratio $S_0(u;\mathbb{R}^n)$ is invariant under scaling $u \mapsto u_R(x) = u(Rx)$; that is, we have

$$S_0(u;\mathbb{R}^n) = S_0(u_R;\mathbb{R}^n), \qquad \text{for all } R > 0 \ . \tag{1.4}$$

In particular, as we already observed in Remark I.4.5, we have:

1.1 Lemma. *$S(\Omega) = S$ is independent of Ω.*

Recalling Remark I.4.7, we note that Lemma 1.1 implies:

1.2 Theorem. *S is never attained on a domain $\Omega \subseteq \mathbb{R}^n,\ \Omega \neq \mathbb{R}^n$.*

Hence, for $\lambda = 0$, the proof of Theorem I.2.1 necessarily fails in the limit case $p = 2^*$. More generally, we have the following uniqueness result, due to Pohožaev [1]:

1.3 Theorem. *Suppose $\Omega \neq \mathbb{R}^n$ is a smooth (possibly unbounded) domain in $\mathbb{R}^n,\ n \geq 3$, which is strictly star-shaped with respect to the origin in $\mathbb{R}^n$, and let $\lambda \leq 0$. Then any solution $u \in H_0^{1,2}(\Omega)$ of the boundary value problem (1.1), (1.3) vanishes identically.*

Theorem 1.3 applies to any solution, whereas Theorem 1.2 is limited to minima of $S_0(\cdot;\Omega)$. However, Theorem 1.2 applies to any domain.

The proof of Theorem 1.3 is based on the following "Pohožaev identity":

1.4 Lemma. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous with primitive $G(u) = \int_0^u g(v)\,dv$ and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the equation*

$$-\Delta u = g(u) \qquad \text{in } \Omega \tag{1.5}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{1.6}$$

in a domain $\Omega \subset\subset \mathbb{R}^n$. Then there holds

$$\frac{n-2}{2}\int_\Omega |\nabla u|^2\,dx - n\int_\Omega G(u)\,dx + \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 x\cdot\nu\,do = 0\ ,$$

where ν denotes the exterior unit normal.

Proof of Theorem 1.3. Let $g(u) = \lambda u + u|u|^{2^*-2}$ with primitive

$$G(u) = \frac{\lambda}{2}|u|^2 + \frac{1}{2^*}|u|^{2^*}\ .$$

By Theorems B.1 and B.2 and Lemma B.3 of Appendix B, any solution of (1.1), (1.3) is smooth on $\overline{\Omega}$. Hence from Pohožaev's identity we infer that

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx - 2^*\int_\Omega G(u)\,dx + \frac{1}{n-2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 x\cdot\nu\,do \\ = \int_\Omega \left(|\nabla u|^2 - |u|^{2^*}\right)dx + \frac{n|\lambda|}{n-2}\int_\Omega |u|^2\,dx \\ + \frac{1}{n-2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 x\cdot\nu\,do = 0\ .\end{aligned}$$

However, testing the Equation (1.1) with u, we infer that

$$\int_\Omega \big(|\nabla u|^2 - \lambda|u|^2 - |u|^{2^*}\big)\, dx = 0\ ,$$

whence

$$2|\lambda| \int_\Omega |u|^2\, dx + \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 x\cdot\nu\, do = 0\ .$$

Moreover, since Ω is strictly star-shaped with respect to $0 \in \mathbb{R}^n$, we have $x\cdot\nu > 0$ for all $x \in \partial\Omega$. Thus $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, and hence $u \equiv 0$ by the principle of unique continuation; see Heinz [2]. □

Proof of Lemma 1.4. Multiply (1.5) by $x\cdot\nabla u$ and compute

$$\begin{aligned} 0 &= \big(\Delta u + g(u)\big)(x\cdot\nabla u) \\ &= \operatorname{div}\big(\nabla u(x\cdot\nabla u)\big) - |\nabla u|^2 - x\cdot\nabla\big(\frac{|\nabla u|^2}{2}\big) + x\cdot\nabla G(u) \\ &= \operatorname{div}\big(\nabla u(x\cdot\nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)\big) + \frac{n-2}{2}|\nabla u|^2 - nG(u)\ . \end{aligned}$$

Upon integrating this identity over Ω and taking account of the fact that by (1.6) we have

$$x\cdot\nabla u = x\cdot\nu\frac{\partial u}{\partial\nu} \qquad \text{on } \partial\Omega\ ,$$

we obtain the asserted identity. □

1.5 Interpretation. Note that the function $x\cdot\nabla u = \frac{d}{dR}u_R$ used in the proof of Lemma 1.4 is the generator of the family of scaled maps $\{u_R\ ;\ 0 < R < \infty\}$. This observation allows a connection between Theorem 1.3 and scale invariance of $S = S_0$. Indeed, we may interpret Theorem 1.3 as reflecting the non-compactness of the multiplicative group $\mathbb{R}_+ = \{R\ ;\ 0 < R < \infty\}$ acting on S via scaling. Note that this group action is a "manifest" symmetry of $S_\lambda(\cdot\,;\Omega)$ only in the case $\lambda = 0$ and $\Omega = \mathbb{R}^n$. In the case of a bounded domain Ω not all scalings $u \to u_R$ will map $H_0^{1,2}(\Omega)$ into itself. For instance, if Ω is an annular region $\Omega = \{x\ ;\ a < |x| < b\}$, in fact, $H_0^{1,2}(\Omega)$ does not admit any of these scalings as symmetries. (In Section 3 we will see that in this case (1.1)–(1.3) does have non-trivial solutions.) However, if Ω is star-shaped with respect to the origin, all scalings $u \to u_R$, $R \geq 1$ will be symmetries of $H_0^{1,2}(\Omega)$, and compactness is lost as $R \to \infty$. The effect is shown in Theorem 1.3.

Note that it is also possible to characterize solutions $u \in H_0^{1,2}(\Omega)$ of equation (1.1) as critical points of a functional E_λ on $H_0^{1,2}(\Omega)$ given by

$$E_\lambda(u) = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 - \lambda|u|^2\big)\, dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\, dx\ . \tag{1.7}$$

By continuity of the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$, the functional E_λ is Fréchet-differentiable on $H_0^{1,2}(\Omega)$. Moreover, for $\lambda < \lambda_1$, the first Dirichlet eigenvalue of the operator $-\Delta$, E_λ satisfies the conditions (1°)–(3°) of the mountain pass lemma Theorem II.6.1; compare the proof of Theorem I.2.1. In view of Theorem II.6.1, the absence of a critical point $u \neq 0$ of E_λ for any $\lambda \leq 0$ proves that E_λ for such λ cannot satisfy the Palais-Smale condition (P.-S.) on a star-shaped domain. Again the non-compact action $R \mapsto u_R(x) = u(Rx)$ can be held responsible.

2. The Brezis-Nirenberg Result

In contrast to Theorem 1.3, for $\lambda > 0$ problem (1.1)–(1.3) may admit non-trivial solutions. However, a subtle dependence on the dimension n is observed.

The first result in this direction is due to Brezis and Nirenberg [2], building on ideas of Trudinger [1] and Aubin [2].

2.1 Theorem. *Suppose Ω is a domain in $\mathbb{R}^n$, $n \geq 3$, and let $\lambda_1 > 0$ denote the first eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions.*
(1°) If $n \geq 4$, then for any $\lambda \in]0, \lambda_1[$ there exists a solution of (1.1)–(1.3).
(2°) If $n = 3$, there exists $\lambda_* \in [0, \lambda_1[$ such that for any $\lambda \in]\lambda_*, \lambda_1[$ problem (1.1)–(1.3) admits a solution.
(3°) If $n = 3$ and $\Omega = B_1(0) \subset \mathbb{R}^3$, then $\lambda_* = \frac{\lambda_1}{4}$ and for $\lambda \leq \frac{\lambda_1}{4}$ there is no solution to (1.1)–(1.3).

As we have seen in Section 1, there are (at least) two different approaches to this theorem. The first, which is the one chosen by Brezis and Nirenberg [2], involves the quotient

$$S_\lambda(u; \Omega) = \frac{\int_\Omega \left(|\nabla u|^2 - \lambda |u|^2\right) dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{2/2^*}} .$$

A second proof can be given along the lines of Theorem II.6.1, applied to the "free" functional E_λ

$$E_\lambda(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2\right) dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx$$

defined earlier. Recall that $E_\lambda \in C^1\left(H_0^{1,2}(\Omega)\right)$. As we shall see, while it is not true that E_λ satisfies the Palais-Smale condition "globally", some compactness will hold in an energy range determined by the best Sobolev constant S; see Lemma 2.3 below. A similar compactness property holds for the functional S_λ.

We will first pursue the approach involving S_λ.

Constrained Minimization

Denote

$$S_\lambda(\Omega) = \inf_{u \in H_0^{1,2}(\Omega)\setminus\{0\}} S_\lambda(u;\Omega) \ .$$

Note that $S_\lambda(\Omega) \le S$ for all $\lambda \ge 0$ (in fact, for all $\lambda \in \mathbb{R}$), and $S_\lambda(\Omega)$ in general is not attained. Similar to Theorem I.4.2 now there holds:

2.2 Lemma. *If Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 3$, and if*

$$S_\lambda(\Omega) < S \ ,$$

then there exists $u \in H_0^{1,2}(\Omega)$, $u > 0$, such that $S_\lambda(\Omega) = S_\lambda(u;\Omega)$.

Proof. Consider a minimizing sequence (u_m) for S_λ in $H_0^{1,2}(\Omega)$. Normalize $\|u_m\|_{L^{2^*}} = 1$. Replacing u_m by $|u_m|$, if necessary, we may assume that $u_m \ge 0$. Since by Hölder's inequality

$$S_\lambda(u_m;\Omega) = \int_\Omega \big(|\nabla u_m|^2 - \lambda |u_m|^2\big)\, dx \ge \int_\Omega |\nabla u_m|^2\, dx - c \ ,$$

we also may assume that $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ as $m \to \infty$.

To proceed, observe that like (I.4.4) by Vitali's convergence theorem we have

$$\begin{aligned}
&\int_\Omega \big(|u_m|^{2^*} - |u_m - u|^{2^*}\big)\, dx \\
&\quad = \int_\Omega \int_0^1 \frac{d}{dt} \big|u_m + (t-1)u\big|^{2^*}\, dt\, dx \\
&\quad = 2^* \int_0^1 \int_\Omega \big(u_m + (t-1)u\big)\big|u_m + (t-1)u\big|^{2^*-2} u\, dx\, dt \\
&\quad \to 2^* \int_0^1 \int_\Omega t\, u |t\, u|^{2^*-2} u\, dx\, dt = \int_\Omega |u|^{2^*}\, dx \quad \text{as } m \to \infty \ .
\end{aligned} \tag{2.1}$$

Also note that

$$\int_\Omega |\nabla u_m|^2\, dx = \int_\Omega |\nabla (u_m - u)|^2\, dx + \int_\Omega |\nabla u|^2\, dx + o(1) \ , \tag{2.2}$$

where $o(1) \to 0$ as $m \to \infty$. Hence we obtain:

$$\begin{aligned}
S_\lambda(u_m;\Omega) + o(1) &= \int_\Omega |\nabla (u_m - u)|^2\, dx + \int_\Omega \big(|\nabla u|^2 - \lambda |u|^2\big)\, dx + o(1) \\
&\ge S \|u_m - u\|_{L^{2^*}}^2 + S_\lambda(\Omega) \|u\|_{L^{2^*}}^2 + o(1) \\
&\ge S \|u_m - u\|_{L^{2^*}}^{2^*} + S_\lambda(\Omega) \|u\|_{L^{2^*}}^{2^*} + o(1) \\
&\ge \big(S - S_\lambda(\Omega)\big) \|u_m - u\|_{L^{2^*}}^{2^*} + S_\lambda(\Omega) + o(1) \ .
\end{aligned}$$

Since $S > S_\lambda(\Omega) = \lim_{m\to\infty} S_\lambda(u_m;\Omega)$ by assumption, we find that $u_m \to u$ in $L^{2^*}(\Omega)$, and $\|u\|_{L^{2^*}} = 1$. By weak lower semi-continuity of the $H_0^{1,2}(\Omega)$-norm then we have

$$S_\lambda(u;\Omega) \le \lim_{m\to\infty} S_\lambda(u_m;\Omega) = S_\lambda(\Omega) \ ,$$

as desired.

Computing the first variation of $S_\lambda(u;\Omega)$, as in the proof of Theorem I.2.1 we see that a positive multiple of u satisfies (1.1), (1.3). Since $u \ge 0,\ u \ne 0$, from the strong maximum principle (Theorem B.4 of Appendix B) we infer that $u > 0$ in Ω. The proof is complete. □

The Unconstrained Case: Local Compactness

Postponing the complete proof of Theorem 2.1 for a moment, we now also indicate the second approach, based on a careful study of the compactness properties of the free functional E_λ. Note that in the case of Theorem 2.1 both approaches are completely equivalent – and the final step in the proof of Theorem 2.1 actually is identical in both cases. However, for more general nonlinearities with critical growth it is not always possible to reduce a boundary value problem like (II.6.1), (II.6.2) to a constrained minimization problem and we will have to use the free functional instead. Moreover, this second approach will bring out the peculiarities of the limiting case more clearly. Our presentation follows Cerami-Fortunato-Struwe [1]. An indication of Lemma 2.3 below is also given by Brezis-Nirenberg [2; p. 463].

2.3 Lemma. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 3$, and let $\lambda \in \mathbb{R}$. Then any sequence (u_m) in $H_0^{1,2}(\Omega)$ such that*

$$E_\lambda(u_m) \to \beta < \frac{1}{n} S^{n/2}, \qquad DE_\lambda(u_m) \to 0 \ ,$$

as $m \to \infty$, is relatively compact.

Proof. To show boundedness of (u_m), compute

$$\begin{aligned} o(1)\big(1 + \|u_m\|_{H_0^{1,2}}\big) + \frac{2}{n} S^{n/2} &\ge 2E_\lambda(u_m) - \langle u_m, DE_\lambda(u_m)\rangle \\ &= \left(1 - \frac{2}{2^*}\right) \int_\Omega |u_m|^{2^*}\, dx \ge c \left(\int_\Omega |u_m|^2\, dx\right)^{2^*/2} , \end{aligned}$$

where $c > 0$ and $o(1) \to 0$ as $m \to \infty$. Hence

$$\begin{aligned} \|u_m\|^2_{H_0^{1,2}} &= 2E_\lambda(u_m) + \lambda \int_\Omega |u_m|^2\, dx + \frac{2}{2^*} \int_\Omega |u_m|^{2^*}\, dx \\ &\le C + o(1)\|u_m\|_{H_0^{1,2}} \ , \end{aligned}$$

and it follows that (u_m) is bounded.

Hence we may assume that $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega)$, and therefore also strongly in $L^p(\Omega)$ for all $p < 2^*$ by the Rellich-Kondrakov theorem; see Theorem A.5 of Appendix A.

In particular, for any $\varphi \in C_0^\infty(\Omega)$ we obtain that

$$\begin{aligned}\langle \varphi, DE_\lambda(u_m)\rangle &= \int_\Omega \left(\nabla u_m \nabla\varphi - \lambda u_m\varphi - u_m|u_m|^{2^*-2}\varphi\right) dx \\ &\to \int_\Omega \left(\nabla u\nabla\varphi - \lambda u\varphi - u|u|^{2^*-2}\varphi\right) dx = \langle \varphi, DE_\lambda(u)\rangle = 0 \ ,\end{aligned}$$

as $m \to \infty$. Hence, $u \in H_0^{1,2}(\Omega)$ weakly solves (1.1). Moreover, choosing $\varphi = u$, we have

$$0 = \langle u, DE_\lambda(u)\rangle = \int_\Omega \left(|\nabla u|^2 - \lambda|u|^2 - |u|^{2^*}\right) dx \ ,$$

and hence

$$E_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega |u|^{2^*}\, dx = \frac{1}{n}\int_\Omega |u|^{2^*}\, dx \geq 0 \ .$$

To proceed, note that by (2.1) and (2.2) we have

$$\begin{aligned}\int_\Omega |\nabla u_m|^2\, dx &= \int_\Omega |\nabla(u_m - u)|^2\, dx + \int_\Omega |\nabla u|^2\, dx + o(1) \ , \\ \int_\Omega |u_m|^{2^*}\, dx &= \int_\Omega |u_m - u|^{2^*}\, dx + \int_\Omega |u|^{2^*}\, dx + o(1) \ ,\end{aligned}$$

where $o(1) \to 0$ $(m \to \infty)$. Hence

$$E_\lambda(u_m) = E_\lambda(u) + E_0(u_m - u) + o(1).$$

Similarly, again using (2.1) we have

$$\begin{aligned}&\int_\Omega u_m|u_m|^{2^*-2}(u_m - u)\, dx \\ &\quad = \int_\Omega \left(|u_m|^{2^*} - |u|^{2^*}\right) dx + o(1) = \int_\Omega |u_m - u|^{2^*}\, dx + o(1) \ ,\end{aligned}$$

whence

$$\begin{aligned}o(1) &= \langle u_m - u, DE_\lambda(u_m)\rangle = \langle u_m - u, DE_\lambda(u_m) - DE_\lambda(u)\rangle \\ &= \int_\Omega \left(|\nabla(u_m - u)|^2 - |u_m - u|^{2^*}\right) dx + o(1) \ .\end{aligned}$$

In particular, from the last equation it follows that

$$E_0(u_m - u) = \frac{1}{n}\int_\Omega |\nabla(u_m - u)|^2\, dx + o(1).$$

On the other hand we have the bound

$$E_0(u_m - u) = E_\lambda(u_m) - E_\lambda(u) + o(1)$$
$$\leq E_\lambda(u_m) + o(1) \leq c < \frac{1}{n} S^{n/2} \qquad \text{for } m \geq m_0 \ .$$

Therefore

$$\|u_m - u\|^2_{H_0^{1,2}} \leq c < S^{n/2} \qquad \text{for } m \geq m_0 \ .$$

But then Sobolev's inequality $\|u_m - u\|^2_{L^{2^*}} \leq S^{-1} \|u_m - u\|^2_{H_0^{1,2}}$ yields the estimate

$$\|u_m - u\|^2_{H_0^{1,2}} \left(1 - S^{-2^*/2} \|u_m - u\|^{2^*-2}_{H_0^{1,2}}\right)$$
$$\leq \int_\Omega \left(|\nabla(u_m - u)|^2 - |u_m - u|^{2^*}\right) dx = o(1) \ ,$$

showing that $u_m \to u$ strongly in $H_0^{1,2}(\Omega)$, as desired. □

Lemma 2.3 motivates one to introduce the following variant of (P.-S.), which seems to appear first in Brezis-Coron-Nirenberg [1].

2.4 Definition. *Let V be a Banach space, $E \in C^1(V)$, $\beta \in \mathbb{R}$. We say that E satisfies condition (P.-S.)$_\beta$, if any sequence (u_m) in V such that $E(u_m) \to \beta$ while $DE(u_m) \to 0$ as $m \to \infty$ is relatively compact. (Such sequences in the sequel, for brevity, will be referred to as (P.-S.)$_\beta$-sequences.)*

In terms of this definition, Lemma 2.3 simply says that the functional E_λ satisfies (P.-S.)$_\beta$ for any $\beta < \frac{1}{n} S^{n/2}$. Now recall that E_λ for $\lambda < \lambda_1$ satisfies conditions (1°)–(3°) of Theorem II.6.1.

By Lemma 2.3, therefore, the proof of the first two parts of Theorem 2.1 will be complete if we can show that for $\lambda > 0$ (respectively $\lambda > \lambda_*$) there holds

$$\beta = \inf_{p \in P} \sup_{u \in p} E_\lambda(u) < \frac{1}{n} S^{n/2} \ , \tag{2.3}$$

where, for a suitable function u_1 satisfying $E_\lambda(u_1) \leq 0$, we let

$$P = \{p \in C^0\big([0,1] \ ; \ H_0^{1,2}(\Omega)\big) \ ; \ p(0) = 0, \ p(1) = u_1\} \ ,$$

as in Theorem II.6.1.

Of course, (2.3) and the condition $S_\lambda(\Omega) < S$ of Lemma 2.2 are related. Given $u \in H_0^{1,2}(\Omega)$ with $\|u\|_{L^{2^*}} = 1$, we may let $p(t) = t\,u$, $u_1 = t_1\,u$ for sufficiently large t_1 to obtain

$$\beta \leq \sup_{0 \leq t < \infty} E_\lambda(t\,u) = \sup_{0 \leq t < \infty} \left(\frac{t^2}{2} S_\lambda(u;\Omega) - \frac{t^{2^*}}{2^*}\right) = \frac{1}{n} S_\lambda^{n/2}(u;\Omega) \ .$$

Hence $\beta \leq \frac{1}{n} S_\lambda^{n/2}(\Omega)$. Likewise, any $p \in P$ contains some $u \neq 0$ such that

$$\langle u, DE_\lambda(u)\rangle = \int_\Omega \left(|\nabla u|^2 - \lambda|u|^2 - |u|^{2^*}\right)\, dx = 0\ .$$

Indeed, since $\lambda < \lambda_1$, for $u = p(t) \neq 0$ with t close to 0 we have $\langle u, DE_\lambda(u)\rangle > 0$, while for $u_1 = p(1)$ we have

$$\langle u_1, DE_\lambda(u_1)\rangle < 2E_\lambda(u_1) \leq 0\ ,$$

and by the intermediate value theorem there exists u, as claimed. But for such u we easily compute

$$\begin{aligned} S_\lambda(\Omega) \leq S_\lambda(u;\Omega) &= \left(\int_\Omega |\nabla u|^2 - \lambda|u|^2\, dx\right)^{1-2/2^*} \\ &= \left(n\, E_\lambda(u)\right)^{2/n} \leq \left(n \sup_{u\in p} E_\lambda(u)\right)^{2/n}\ . \end{aligned}$$

Upon passing to the infimum with respect to $p \in P$ we find that

$$\frac{1}{n} S_\lambda^{n/2}(\Omega) \leq \beta = \inf_{p\in P} \sup_{u\in p} E_\lambda(u) \leq \frac{1}{n} S_\lambda^{n/2}(\Omega)\ ; \tag{2.4}$$

that is, (2.3) and the condition $S_\lambda < S$ are in fact equivalent.

Proof of Theorem 2.1(1°). It suffices to show that $S_\lambda < S$. Consider the family

$$u_\varepsilon^*(x) = \frac{\left[n(n-2)\varepsilon^2\right]^{\frac{n-2}{4}}}{\left[\varepsilon^2 + |x|^2\right]^{\frac{n-2}{2}}},\ \varepsilon > 0\ , \tag{2.5}$$

of functions $u_\varepsilon^* \in D^{1,2}(\mathbb{R}^n)$. Note that $u_\varepsilon^*(x) = \varepsilon^{\frac{2-n}{2}} u_1^*\left(\frac{x}{\varepsilon}\right)$, and u_ε^* satisfies the equation

$$-\Delta u_\varepsilon^* = u_\varepsilon^* |u_\varepsilon^*|^{2^*-2} \qquad \text{in } \mathbb{R}^n\ , \tag{2.6}$$

as is easily verified by a direct computation. We claim that $S_0\big(u_\varepsilon^*; \mathbb{R}^n\big) = S$; that is, the best Sobolev constant is achieved by the family u_ε^*, $\varepsilon > 0$. Indeed, let $u \in D^{1,2}(\mathbb{R}^n)$ satisfy $S_0(u;\mathbb{R}^n) = S$. (The existence of such a function u can be deduced for instance from Theorem I.4.9.) Using Schwarz-symmetrization we may assume that u is radially symmetric; that is, $u(x) = u(|x|)$. (In fact, any positive solution of (2.6), which decays to 0 sufficiently rapidly as $|x| \to \infty$, by the result of Gidas-Ni-Nirenberg quoted in Section I.2.2 above is radially symmetric.) Moreover, u solves (2.6). Choose $\varepsilon_0 > 0$ such that $u_{\varepsilon_0}^*(0) = u(0)$. Then u and $u_{\varepsilon_0}^*$ both are solutions of the ordinary differential equation of second order in $r = |x|$,

$$r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) u = u|u|^{2^*-2} \qquad \text{for } r > 0\ ,$$

sharing the initial data $u(0) = u_{\varepsilon_0}^*(0)$, $\partial_r u(0) = \partial_r u_{\varepsilon_0}^*(0) = 0$. It is not hard to prove that this initial value problem admits a unique solution. Thus $u = u_{\varepsilon_0}^*$, which implies that $S = S_0(u;\mathbb{R}^n) = S_0(u_{\varepsilon_0}^*;\mathbb{R}^n) = S_0(u_\varepsilon^*;\mathbb{R}^n)$ for all $\varepsilon > 0$.

Since (2.6) also yields that $\|u_\varepsilon^*\|^2_{H_0^{1,2}} = \|u_\varepsilon^*\|^{2^*}_{L^{2^*}}$ it follows that

$$\|u_\varepsilon^*\|^2_{H_0^{1,2}} = \|u_\varepsilon^*\|^{2^*}_{L^{2^*}} = S^{n/2} \text{ for all } \varepsilon > 0 \,.$$

We may suppose that $0 \in \Omega$. Let $\eta \in C_0^\infty(\Omega)$ be a fixed cut-off function, $\eta \equiv 1$ in a neighborhood $B_\rho(0)$ of 0. Let $u_\varepsilon = \eta\, u_\varepsilon^*$ and compute

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^2\, dx &= \int_\Omega |\nabla u_\varepsilon^*|^2 \eta^2\, dx + O\big(\varepsilon^{n-2}\big) \\
&= \int_{\mathbb{R}^n} |\nabla u_\varepsilon^*|^2\, dx + O\big(\varepsilon^{n-2}\big) = S^{n/2} + O\big(\varepsilon^{n-2}\big)\,. \\
\int_\Omega |u_\varepsilon|^{2^*}\, dx &= \int_{\mathbb{R}^n} |u_\varepsilon^*|^{2^*}\, dx + O\big(\varepsilon^n\big) = S^{n/2} + O\big(\varepsilon^n\big) \\
\int_\Omega |u_\varepsilon|^2\, dx &\geq \int_{B_\rho(0)} |u_\varepsilon^*|^2\, dx \\
&\geq \int_{B_\varepsilon(0)} \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{2}}}{[2\varepsilon^2]^{n-2}}\, dx \qquad (2.7) \\
&\quad + \int_{B_\rho(0)\setminus B_\varepsilon(0)} \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{2}}}{[2|x|^2]^{n-2}}\, dx \\
&= c_1 \cdot \varepsilon^2 + c_2 \varepsilon^{n-2} \int_\varepsilon^\rho r^{3-n}\, dr \\
&= \begin{cases} c\,\varepsilon^2 + O\big(\varepsilon^{n-2}\big), & \text{if } n > 4 \\ c\,\varepsilon^2 |\ln \varepsilon| + O\big(\varepsilon^2\big), & \text{if } n = 4 \\ c\,\varepsilon + O\big(\varepsilon^2\big), & \text{if } n = 3 \end{cases}
\end{aligned}$$

with positive constants $c, c_1, c_2 > 0$. Thus, if $n \geq 5$

$$\begin{aligned}
S_\lambda(u_\varepsilon; \Omega) &\leq \frac{(S^{n/2} - c\lambda\varepsilon^2 + O(\varepsilon^{n-2}))}{(S^{n/2} + O(\varepsilon^n))^{2/2^*}} \\
&= S - c\lambda\varepsilon^2 + O\big(\varepsilon^{n-2}\big) < S\,,
\end{aligned}$$

if $\varepsilon > 0$ is sufficiently small. Similarly, if $n = 4$, we have

$$S_\lambda(u_\varepsilon; \Omega) \leq S - c\lambda\varepsilon^2 |\ln \varepsilon| + O\big(\varepsilon^2\big) < S$$

for $\varepsilon > 0$ sufficiently small.

Remark on Theorem 2.1(2°),(3°). If $n = 3$, estimate (2.7) shows that the "gain" due to the presence of λ and the "loss" due to truncation of u_ε^* may be of the same order in ε; hence S_λ can only be expected to be smaller than S for "large" λ. To see that $\lambda_* < \lambda_1$, choose the first eigenfunction $u = \varphi_1$ of $(-\Delta)$ as comparison function. The non-existence result for $\Omega = B_1(0)$, $\lambda \leq \frac{\lambda_1}{4}$ follows from a weighted estimate similar to Lemma 1.4; see Brezis-Nirenberg [2; Lemma 1.4]. We omit the details. □

Theorem 2.1 should be viewed together with the global bifurcation result of Rabinowitz [1; p. 195 f.]. Intuitively, Theorem 2.1 indicates that the branch of positive solutions found by Rabinowitz in dimension $n \geq 4$ on a star-shaped domain bends back to $\lambda = 0$ and becomes asymptotic to this axis.

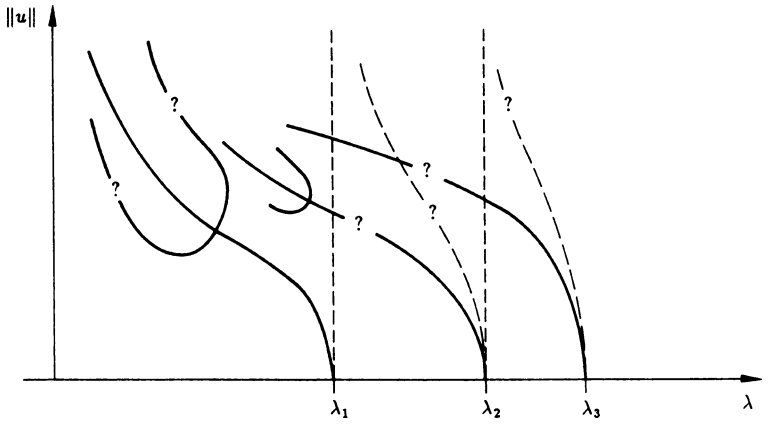

Fig. 2.1. Solution "branches" for (1.1), (1.3) depending on λ

However, Equation (1.1) may have many positive solutions. In Section 3 we shall see that if Ω is an annulus, positive radial solutions exist for any value of $\lambda < \lambda_1$. But note that, by Theorem 1.2, for $\lambda = 0$ these cannot minimize S_λ. Hence we may have many different branches of (or secondary bifurcations of branches of) positive solutions, in general.

Multiple Solutions

2.5 Bifurcation from higher eigenvalue. If we drop the requirement of positivity, the local Palais-Smale condition Lemma 2.3 permits us to obtain bifurcation of non-trivial solutions of (1.1), (1.3) from higher eigenvalues, as well. Recall that by a result of Böhme [1] and Marino [1] it is known that any eigenvalue of $-\Delta$ on $H_0^{1,2}(\Omega)$ is a point of bifurcation from the trivial solution of (1.1), (1.3). However, the variational method may give better estimates for the λ-interval of existence for such solutions.

The following result is due to Cerami-Fortunato-Struwe [1]:

2.6 Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$, and let $0 < \lambda_1 < \lambda_2 \leq \ldots$ denote the eigenvalues of $-\Delta$ in $H_0^{1,2}(\Omega)$. Also let*

$$\nu = S \cdot \left(\mathcal{L}^n(\Omega)\right)^{-2/n} > 0 \ .$$

Then, if

$$m = m(\lambda) = \sharp\{j \ ; \ \lambda < \lambda_j < \lambda + \nu\} \ ,$$

problem (1.1), (1.3) admits at least m distinct pairs of non-trivial solutions.

Remark. From the Weyl formula $\lambda_j \sim C(\Omega) j^{2/n}$ for the asymptotic behavior of the eigenvalues λ_j we conclude that $m(\lambda) \to \infty$ as $\lambda \to \infty$.

Proof of Theorem 2.6. Consider the functional

$$S_\lambda(u) := S_\lambda(u;\Omega) = \frac{\int_\Omega \big(|\nabla u|^2 - \lambda |u|^2\big)\, dx}{\big(\int_\Omega |u|^{2^*}\, dx\big)^{2/2^*}}$$

on the unit sphere

$$M = \{u \in H_0^{1,2}(\Omega) \ ; \ \|u\|_{L^{2^*}} = 1\} \ .$$

$M \subset H_0^{1,2}(\Omega)$ is a complete Hilbert manifold, invariant under the involution $u \to -u$. Recall that S_λ is differentiable on M and that if $u \in M$ is a critical point of S_λ with $S_\lambda(u) = \beta > 0$, then $\tilde{u} = \beta^{\frac{1}{2^*-2}} u$ solves (1.1), (1.3) with

$$E_\lambda(\tilde{u}) = \frac{1}{n} S_\lambda(u)^{n/2} \ ;$$

see the proof of Theorem I.2.1.

Moreover, (u_m) is a (P.-S.)$_\beta$-sequence for S_λ if and only if the sequence $(\tilde{u}_m)$, where $\tilde{u}_m = \beta^{\frac{1}{2^*-2}} u_m$, is a (P.-S.)$_{\tilde{\beta}}$ sequence for E_λ with $\tilde{\beta} = \frac{1}{n}\beta^{n/2}$. In particular, by Lemma 2.3 we have that S_λ satisfies (P.-S.)$_\beta$ on M for any $\beta \in]0, S[$.

Now let γ denote the Krasnoselskii genus, and for $j \in \mathbb{N}$ such that $\lambda_j \in]\lambda, \lambda + \nu[$ let

$$\beta_j = \inf_{\substack{A \subset M \\ \gamma(A) \geq j}} \sup_{u \in A} S_\lambda(u) \ .$$

Note that by Proposition II.5.3 for any $A \subset M$ such that $\gamma(A) \geq j$ there exist at least j mutually orthogonal vectors in A, whence $\beta_j > 0$ for all j as above. Moreover, if we denote by $\varphi_k \in H_0^{1,2}(\Omega)$ the kth eigenfunction of $-\Delta$ and let $A_j = \operatorname{span}\{\varphi_1, \ldots, \varphi_j\} \cap M$, we obtain that

$$\begin{aligned} \beta_j &\leq \sup_{u \in A_j} S_\lambda(u) \leq (\lambda_j - \lambda) \sup_{u \in A_j} \frac{\int_\Omega |u|^2\, dx}{\big(\int_\Omega |u|^{2^*}\, dx\big)^{2/2^*}} \\ &< \nu\big(\mathcal{L}^n(\Omega)\big)^{\frac{2}{n}} = S \ . \end{aligned}$$

Hence the theorem follows from Theorem II.4.2 and Lemma II.5.6. □

We do not know how far the solution branches bifurcating off the trivial solution $u \equiv 0$ at $\lambda = \lambda_j$ extend. However, the following result has been obtained by Capozzi-Fortunato-Palmieri [1]:

2.7 Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 4$. Then for any $\lambda > 0$ problem (1.1), (1.3) admits a non-trivial solution.*

The proof of Capozzi-Fortunato-Palmieri is based on a linking argument and the local Palais-Smale condition Lemma 2.3. A simpler proof for λ not belonging to the spectrum of $-\Delta$, using the duality method, was worked out by Ambrosetti and Struwe [1].

Theorem 2.7 leaves open the question of multiplicity. For $n \geq 4$, $\lambda > 0$ Fortunato-Jannelli [1] present results in this direction using symmetries of the domain to restrict the space of admissible functions to certain symmetric subspaces, where the first Dirichlet eigenvalue is increased sufficiently for Theorem 2.1 to become applicable.

For instance, if $\Omega = B_1(0;\mathbb{R}^2) \times \Omega' \subset \mathbb{R}^n$, $n \geq 4$, $\lambda > 0$, essentially Fortunato-Jannelli solve (1.1), (1.3) by constructing a positive solution u to (1.1), (1.3) on a "slice"

$$\Omega_m = \{x = re^{i\varphi} \in B_1(0;\mathbb{R}^2)\ ;\ 0 \leq \varphi \leq \frac{\pi}{m}\} \times \Omega'$$

and reflecting u in the "vertical" edges of Ω_m a total of $2m$ times. Since the first eigenvalue $\lambda_1^{(m)}$ of $(-\Delta)$, acting on $H_0^{1,2}(\Omega_m)$, tends to ∞ as $m \to \infty$, we can achieve that $\lambda_1^{(m)} > \lambda$ for $m \geq m_0$. Hence the existence of a positive solution u to (1.1), (1.3) on Ω_m is guaranteed by Theorem 2.1.

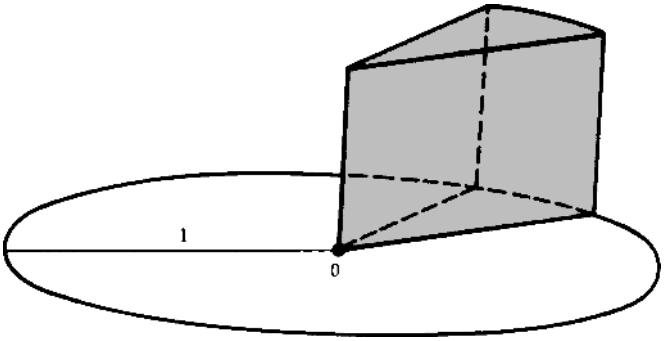

Fig. 2.2. A slice Ω_m of the pie Ω

A different idea is used by Ding [1] to construct infinitely many solutions of varying sign of Equation (2.6) on $\mathbb{R}^n$. In fact, Ding's result involves a beautiful combination of analysis and geometry. Conformally mapping $\mathbb{R}^n$ to the n-sphere and imposing invariance with respect to a subgroup $SO(k) \subset SO(n)$ of rotations of the sphere as a constraint on admissible comparison functions,

Ding effectively reduces Equation (2.6) to a sub-critical variational problem to which Theorem 4.2 may be applied.

For the existence of multiple radially symmetric solutions on balls $\Omega = B_R(0) \subset \mathbb{R}^n$ a subtle dependence on the space dimension is observed, related to the fact that the best Sobolev constant S is attained in the class of radially symmetric functions (on $\mathbb{R}^n$). The following result is due to Cerami-Solimini-Struwe [1] and Solimini [1] :

2.8 Theorem. *Suppose $\Omega = B_R(0)$ is a ball in $\mathbb{R}^n, n \geq 7$. Then for any $\lambda > 0$ problem (1.1), (1.3) admits infinitely many radially symmetric solutions.*

The proof of Theorem 2.8 uses a characterization of different solutions by the nodal properties they possess, as in the locally compact case; see Remark II.7.3.

By recent results of Atkinson-Brezis-Peletier [1] the restriction on the dimension n in Theorem 2.8 appears to be sharp; see also Adimurthi-Yadava [1].

In 2002, finally, Devillanova-Solimini [1] showed that for any $\lambda > 0$ problem (1.1), (1.3) admits infinitely many solutions also on an arbitrary smoothly bounded domain $\Omega \subset \mathbb{R}^n$, provided that $n \geq 7$. Their method of proof relies on a precise analysis as $p \uparrow 2^*$ of solutions $u = u_p$ to the sub-critical equations

$$-\Delta u = \lambda u + u|u|^{p-2} \quad \text{in } \Omega \ ,$$

with $u = 0$ on $\partial\Omega$. Again their method indicates that the restriction $n \geq 7$ may be sharp.

2.9 Notes. (1°) Egnell [1], Guedda-Veron [1], and Lions-Pacella-Tricarico [1] have studied problems of the type (1.1), (1.3) involving the (degenerate) pseudo-Laplace operator and partially free boundary conditions.
(2°) The linear term λu in Equation (1.1) may be replaced by other compact perturbations; see Brezis-Nirenberg [2]. In this regard we also mention the results by Mancini-Musina [1] concerning obstacle problems of type (1.1), (1.3). Still a different kind of "perturbation" will be considered in the next section.

3. The Effect of Topology

Instead of a star-shaped domain, as in Theorem 1.3, consider an annulus

$$\Omega = \{x \in \mathbb{R}^n \ ; \ r_1 < |x| < r_2\} \ ,$$

and for $\lambda \in \mathbb{R}$ let E_λ be given by (1.7). Note that if we restrict our attention to radial functions

$$H^{1,2}_{0,rad}(\Omega) = \{u \in H^{1,2}_0(\Omega) \ ; \ u(x) = u(|x|)\} \ ,$$

by estimate (II.11.6) the embedding

$$H^{1,2}_{0,rad}(\Omega) \hookrightarrow L^p(\Omega)$$

is compact for any $p < \infty$. Hence $DE_\lambda \colon H^{1,2}_{0,rad}(\Omega) \to H^{-1}(\Omega)$ is of the form *id + compact*. Since any (P.-S.)-sequence for E_λ by the proof of Lemma 2.3 is bounded, from Proposition II.2.2 we thus infer that E_λ satisfies (P.-S.) globally on $H^{1,2}_{0,rad}(\Omega)$ and hence from Theorem II.5.7 or Theorem II.6.5 that problem (1.1), (1.3) possesses infinitely many radially symmetric solutions on Ω, for any $n \geq 2$, and any $\lambda \in \mathbb{R}$, in particular, for $\lambda = 0$. This result stands in striking contrast with Theorem 1.3 (or Theorem 2.8, as regards the restriction of the dimension).

In the following we shall investigate whether the solvability of (1.1), (1.3) on an annulus is a singular phenomenon, observable only in a highly symmetric case, or is stable and survives perturbations of the domain.

A Global Compactness Result

Remark that by Theorem 1.2, for $\lambda = 0$ no non-trivial solution $u \in H^{1,2}_0(\Omega)$ of (1.1) can satisfy $S_\lambda(u;\Omega) \leq S$. Hence the local compactness of Lemma 2.3 will not suffice to produce such solutions and we must study the compactness properties of E_λ, respectively S_λ, at higher energy levels as well. The next result can be viewed as an extension of P.-L. Lions' concentration-compactness principle for minimization problems (see Section I.4) to problems of minimax type. The idea of analyzing the behavior of a (P.-S.)-sequence near points of concentration by "blowing up" the singularities seems to appear first in papers by Sacks-Uhlenbeck [1] and Wente [5], where variants of the local compactness condition Lemma 2.3 are obtained (see Sacks-Uhlenbeck [1; Lemma 4.2]). In the next result, due to Struwe [8], we systematically employ the blow-up technique to characterize *all* energy values β of a variational problem where (P.-S.)$_\beta$ may fail.

3.1 Theorem. *Suppose Ω is a smoothly bounded domain in $\mathbb{R}^n, n \geq 3$, and for $\lambda \in \mathbb{R}$ let (u_m) be a (P.-S.)-sequence for E_λ in $H^{1,2}_0(\Omega) \subset D^{1,2}(\mathbb{R}^n)$. Then there exist a number $k \in \mathbb{N}_0$, sequences (R^j_m), (x^j_m), $1 \leq j \leq k$, of radii $R^j_m \to \infty$ $(m \to \infty)$ and points $x^j_m \in \Omega$, a solution $u^0 \in H^{1,2}_0(\Omega) \subset D^{1,2}(\mathbb{R}^n)$ to (1.1), (1.3) and non-trivial solutions $u^j \in D^{1,2}(\mathbb{R}^n)$, $1 \leq j \leq k$, to the "limiting problem" associated with (1.1) and (1.3),*

$$-\Delta u = u|u|^{2^*-2} \qquad \text{in } \mathbb{R}^n \ , \tag{3.1}$$

such that a subsequence (u_m) satisfies

$$\Big\| u_m - u^0 - \sum_{j=1}^{k} u^j_m \Big\|_{D^{1,2}(\mathbb{R}^n)} \to 0 \ .$$

Here u^j_m denotes the rescaled function

$$u_m^j(x) = (R_m^j)^{\frac{n-2}{2}} u^j\big(R_m^j(x - x_m^j)\big),\ 1 \le j \le k,\ m \in \mathbb{N}\ .$$

Moreover, we have

$$E_\lambda(u_m) \to E_\lambda(u^0) + \sum_{j=1}^{k} E_0(u^j)\,.$$

3.2 Remarks. (1°) In particular, if Ω is a ball $\Omega = B_R(0), u_m \in H^{1,2}_{0,rad}(\Omega)$, from the uniqueness of the family $(u_\varepsilon^*)_{\varepsilon>0}$ of radial solutions to (3.1) – see the proof of Theorem 2.1(1°) – it follows that each u^j is of the form (2.5) with $E_0(u^j) = \frac{1}{n}S^{n/2} =: \beta^*$. Hence in this case (P.-S.)$_\beta$ holds for E_λ for all levels β which cannot be decomposed

$$\beta = \beta_0 + k\beta^*\ ,$$

where $k \ge 1$ and $\beta_0 = E_\lambda(u^0)$ is the energy of some radial solution of (1.1), (1.3). Similarly, if Ω is an arbitrary bounded domain and $u_m \ge 0$ for all m, then also $u^j \ge 0$ for all j, and by a result of Gidas-Ni-Nirenberg [1; p. 210 f.] and Obata [1] again each function u^j will be radially symmetric about some point x^j. Therefore also in this case each u^j is of the form $u^j = u_\varepsilon^*(\cdot - x^j)$ for some $\varepsilon > 0$, and (P.-S.)$_\beta$ holds for all β which are not of the form $\beta = \beta_0 + k\beta^*$ where $k \ge 1$ and $\beta_0 = E_\lambda(u_0)$ is the energy of some non-negative solution u^0 of (1.1), (1.3).
(2°) For some time it was believed that the family (2.5) gives all non-trivial solutions of (3.1). However, Ding's [1] result shows that (3.1) also admits infinitely many solutions of changing sign which are distinct modulo scaling.
(3°) In general, decomposing a solution v of (3.1) into positive and negative parts $v = v_+ + v_-$, where $v_\pm = \pm\max\{\pm v, 0\}$, upon testing (3.1) with $v_\pm$ from Sobolev's inequality we infer that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \big(-\Delta v - v|v|^{2^*-2}\big)v_\pm\, dx \\ &= \int_{\mathbb{R}^n} \big(|\nabla v_\pm|^2 - |v_\pm|^{2^*}\big)\, dx \ge \Big(1 - S^{-2^*/2}\|v_\pm\|_{D^{1,2}}^{2^*-2}\Big)\,\|v_\pm\|_{D^{1,2}}^2\ . \end{aligned}$$

Hence, either $v \equiv 0$, or

$$E_0(v_\pm) = \frac{1}{n}\|v_\pm\|_{D^{1,2}}^2 \ge \frac{1}{n}S^{n/2} = \beta^*.$$

In fact, $E_0(v_\pm) > \beta^*$; otherwise S would be achieved at $v_\pm$, contradicting Theorem 1.2. Therefore any solution v of (3.1) that changes sign satisfies

$$E_0(v) = E_0(v_+) + E_0(v_-) > 2\beta^*,$$

and in Theorem 3.1 we can assert that $E_0(u^j) \in \{\beta^*\} \cup\,]2\beta^*, \infty[$.

In particular, if (1.1), (1.3) does not admit any solution but the trivial solution $u \equiv 0$, the local Palais-Smale condition (P.-S.)$_\beta$ will hold for all $\beta < 2\beta^*$, except for $\beta = \beta^*$.

When $\lambda = 0$, as noted above, any non-trivial solution u of problem (1.1), (1.3) on a bounded domain satisfies $E_0(u) > \beta^*$. Theorem 3.1 then likewise guarantees that condition (P.-S.)$_\beta$ will hold for all $\beta < 2\beta^*$, except for $\beta = \beta^*$. See Weth [1] for a recent improvement of this result.

(4°) Bahri-Coron [1] observe that in Theorem 3.1 in addition we can assert

$$\frac{R_m^i}{R_m^j} + \frac{R_m^j}{R_m^i} + R_m^i R_m^j |x_m^i - x_m^j|^2 \to \infty \text{ as } m \to \infty \text{ for all } i \neq j.$$

Proof of Theorem 3.1. First recall that as in the proof of Lemma 2.3 any (P.-S.)-sequence for E_λ is bounded. Hence we may assume that $u_m \rightharpoonup u^0$ weakly in $H_0^{1,2}(\Omega)$, and u^0 solves (1.1), (1.3). Moreover, if we let $v_m = u_m - u^0$ we have $v_m \to 0$ strongly in $L^2(\Omega)$, and by (2.1), (2.2) also that

$$\int_\Omega |v_m|^{2^*}\, dx = \int_\Omega |u_m|^{2^*}\, dx - \int_\Omega |u^0|^{2^*}\, dx + o(1)\ ,$$
$$\int_\Omega |\nabla v_m|^2\, dx = \int_\Omega |\nabla u_m|^2\, dx - \int_\Omega |\nabla u^0|^2\, dx + o(1)\ ,$$

where $o(1) \to 0$ $(m \to \infty)$. Hence, in particular, we obtain that

$$E_\lambda(u_m) = E_\lambda(u^0) + E_0(v_m) + o(1)\ .$$

Also note that

$$DE_\lambda(u_m) = DE_\lambda(u^0) + DE_0(v_m) + o(1) = DE_0(v_m) + o(1)\ ,$$

where $o(1) \to 0$ in $H^{-1}(\Omega)$ $(m \to \infty)$. Using the following lemma, we can now proceed by induction:

3.3 Lemma. *Suppose (v_m) is a (P.-S.)-sequence for $E = E_0$ in $H_0^{1,2}(\Omega)$ such that $v_m \rightharpoonup 0$ weakly. Then there exists a sequence (x_m) of points $x_m \in \Omega$, a sequence (R_m) of radii $R_m \to \infty$ $(m \to \infty)$, a non-trivial solution v^0 to the limiting problem (3.1) and a (P.-S.)-sequence (w_m) for E_0 in $H_0^{1,2}(\Omega)$ such that for a subsequence (v_m) there holds*

$$w_m = v_m - R_m^{\frac{n-2}{2}} v^0\big(R_m(\cdot - x_m)\big) + o(1)\ ,$$

where $o(1) \to 0$ in $D^{1,2}(\mathbb{R}^n)$ as $m \to \infty$. In particular, $w_m \rightharpoonup 0$ weakly. Furthermore,

$$E_0(w_m) = E_0(v_m) - E_0(v^0) + o(1)\ .$$

Moreover,

$$R_m\ \mathrm{dist}(x_m, \partial\Omega) \to \infty\ .$$

Finally, if $E_0(v_m) \to \beta < \beta^*$, the sequence (v_m) is relatively compact and hence $v_m \to 0$ strongly in $H_0^{1,2}(\Omega)$, $E_0(v_m) \to \beta = 0$ as $m \to \infty$.

Proof of Theorem 3.1 (completed). Apply Lemma 3.3 to the sequences $v_m^1 = u_m - u^0$, $v_m^j = u_m - u^0 - \sum_{i=1}^{j-1} u_m^i = v_m^{j-1} - u_m^{j-1}$, $j > 1$, where

$$u_m^i(x) = (R_m^i)^{\frac{n-2}{2}} u^i\big(R_m^i(x - x_m^i)\big) \ .$$

By induction

$$\begin{aligned} E_0(v_m^j) &= E_\lambda(u_m) - E_\lambda(u^0) - \sum_{i=1}^{j-1} E_0(u^i) \\ &\le E_\lambda(u_m) - (j-1)\beta^* \ . \end{aligned}$$

Since the latter will be negative for large j, by Lemma 3.3 the induction will terminate after some index $k \ge 0$. Moreover, for this index we have

$$v_m^{k+1} = u_m - u^0 - \sum_{j=1}^{k} u_m^j \to 0$$

strongly in $D^{1,2}(\mathbb{R}^n)$, and

$$E_\lambda(u_m) - E_\lambda(u^0) - \sum_{j=1}^{k} E_0(u^j) \to 0 \ ,$$

as desired. □

Proof of Lemma 3.3. If $E_0(v_m) \to \beta < \beta^*$, by Lemma 2.3 the sequence (v_m) is strongly relatively compact and hence $v_m \to 0$, $\beta = 0$. Therefore, we may assume that $E_0(v_m) \to \beta \ge \beta^* = \frac{1}{n} S^{n/2}$. Moreover, since $DE_0(v_m) \to 0$ we also have

$$\frac{1}{n} \int_\Omega |\nabla v_m|^2 \, dx = E_0(v_m) - \frac{1}{2^*} \langle v_m, DE_0(v_m) \rangle \to \beta \ge \frac{1}{n} S^{n/2}$$

and hence that

$$\liminf_{m \to \infty} \int_\Omega |\nabla v_m|^2 \, dx = n\beta \ge S^{n/2} \ . \tag{3.2}$$

Extend $v_m \equiv 0$ outside Ω. Denote

$$Q_m(r) = \sup_{x \in \Omega} \int_{B_r(x)} |\nabla v_m|^2 \, dx$$

the concentration function of v_m, introduced in Section I.4.3. Choose $x_m \in \mathbb{R}^n$ and scale

$$v_m \mapsto \tilde{v}_m(x) = R_m^{\frac{2-n}{2}} v_m\big(x/R_m + x_m\big)$$

such that

$$\tilde{Q}_m(1) = \sup_{x\in\mathbb{R}^n} \int_{B_1(x)} |\nabla \tilde{v}_m|^2 \, dx = \int_{B_1(0)} |\nabla \tilde{v}_m|^2 \, dx = \frac{1}{2L} S^{n/2} \, ,$$

where L is a number such that $B_2(0)$ is covered by L balls of radius 1. Clearly, by (3.2) we have $R_m \geq R_0 > 0$, uniformly in m.

Letting $\tilde{\Omega}_m = \{x \in \mathbb{R}^n \, ; \, x/R_m + x_m \in \Omega\}$, we may assume that $\tilde{\Omega}_m \to \tilde{\Omega}_\infty \subset \mathbb{R}^n$. Since

$$\|\tilde{v}_m\|^2_{D^{1,2}} = \|v_m\|^2_{D^{1,2}} \to n\beta < \infty,$$

we also may assume that $\tilde{v}_m \rightharpoonup v^0$ weakly in $D^{1,2}(\mathbb{R}^n)$. By density of $C_0^\infty(\tilde{\Omega}_\infty)$ in $D^{1,2}(\tilde{\Omega}_\infty)$, moreover, we can find functions $\tilde{v}^0_m \in H_0^{1,2}(\tilde{\Omega}_m)$ such that $v^0 = \lim_{m\to\infty} \tilde{v}^0_m$ in $D^{1,2}(\tilde{\Omega}_\infty)$.

We claim that $\tilde{v}_m \to v^0$ strongly in $H^{1,2}(\Omega')$, or, equivalently, that the functions $(\tilde{v}_m - \tilde{v}^0_m) \to 0$ strongly in $H^{1,2}(\Omega')$ for any $\Omega' \subset\subset \mathbb{R}^n$. It suffices to consider balls $\Omega' = B_1(x_0)$ for any $x_0 \in \mathbb{R}^n$. For brevity in the following we let $B_r(x_0) =: B_r$.

Choose cut-off functions $\varphi_i \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi_i \leq 1$, $i = 1, 2$, $\varphi_1 \equiv 1$ in B_1, $\varphi_1 \equiv 0$ outside $B_{3/2}$, while $\varphi_2 \equiv 1$ in $B_{3/2}$, $\varphi_2 \equiv 0$ outside B_2. Then let $\tilde{w}^i_m = (\tilde{v}_m - \tilde{v}^0_m)\varphi_i \in H_0^{1,2}(\tilde{\Omega}_m)$, $i = 1, 2$. Note that we have

$$\|\tilde{w}^i_m\|_{H^{1,2}(\tilde{\Omega}_m)} \leq c\|\tilde{v}_m - \tilde{v}^0_m\|_{H^{1,2}(\tilde{\Omega}_m)} \leq C < \infty, \; i = 1, 2.$$

Moreover, $\tilde{w}^1_m$ is supported in the ball $B_{3/2}$, and $\tilde{w}^2_m$ is supported in B_2.

Recall that $(\tilde{v}_m - \tilde{v}^0_m) \rightharpoonup 0$ weakly in $D^{1,2}(\mathbb{R}^n)$. The restrictions of these functions to B_2 then converge weakly to 0 in $H^{1,2}(B_2)$. From compactness of the embedding $H^{1,2}(B_2) \hookrightarrow L^p(B_2)$ for any $p < 2^*$, finally, we also find that $(\tilde{v}_m - \tilde{v}^0_m) \to 0$ strongly in $L^p(B_2)$ for any such p.

Using convergence arguments familiar by now and Sobolev's inequality, we then obtain

$$\begin{aligned}
o(1) &= \langle \tilde{w}^1_m \varphi_1, DE_0(\tilde{v}_m; \tilde{\Omega}_m)\rangle \\
&= \int_{\mathbb{R}^n} \big(\nabla \tilde{v}_m \nabla \tilde{w}^1_m - \tilde{v}_m |\tilde{v}_m|^{2^*-2} \tilde{w}^1_m\big)\varphi_1 \, dx + o(1) \\
&= \int_{\mathbb{R}^n} \big(|\nabla(\tilde{v}_m - \tilde{v}^0_m)|^2 - |\tilde{v}_m - \tilde{v}^0_m|^{2^*}\big)\varphi_1^2 \, dx + o(1) \\
&= \int_{\mathbb{R}^n} \big(|\nabla(\tilde{v}_m - \tilde{v}^0_m)|^2 \varphi_1^2 - |\tilde{v}_m - \tilde{v}^0_m|^{2^*} \varphi_1^2 \varphi_2^{2^*-2}\big) \, dx + o(1) \\
&= \int_{\mathbb{R}^n} \big(|\nabla \tilde{w}^1_m|^2 - |\tilde{w}^1_m|^2 |\tilde{w}^2_m|^{2^*-2}\big) \, dx + o(1) \\
&\geq \|\tilde{w}^1_m\|^2_{D^{1,2}(\mathbb{R}^n)} \Big(1 - S^{-2^*/2} \|\tilde{w}^2_m\|^{2^*-2}_{D^{1,2}(\mathbb{R}^n)}\Big) + o(1) \, ,
\end{aligned} \tag{3.3}$$

where $o(1) \to 0$ as $m \to \infty$. But note that

$$\int_{\mathbb{R}^n} |\nabla \tilde{w}_m^2|^2 \, dx = \int_{\mathbb{R}^n} |\nabla(\tilde{v}_m - \tilde{v}_m^0)|^2 \varphi_2^2 \, dx + o(1)$$
$$= \int_{\mathbb{R}^n} \big(|\nabla \tilde{v}_m|^2 - |\nabla v^0)|^2\big)\varphi_2^2 \, dx + o(1) \le \int_{B_2} |\nabla \tilde{v}_m|^2 \, dx + o(1)$$
$$\le L\, \tilde{Q}_m(1) + o(1) = \frac{1}{2} S^{n/2} + o(1)\,.$$

Thus $S^{-2^*/2}\|\tilde{w}_m^2\|_{D^{1,2}(\mathbb{R}^n)}^{2^*-2} \le c < 1$ for large m, and from (3.3) it follows that $\tilde{w}_m^1 \to 0$ in $D^{1,2}(\mathbb{R}^n)$; that is, $\tilde{v}_m \to v^0$ strongly locally in $H^{1,2}$, as desired.

In particular,

$$\int_{B_1(0)} |\nabla v^0|^2 \, dx = \frac{1}{2L} S^{n/2} > 0\ ,$$

and $v^0 \not\equiv 0$. Since the original sequence $v_m \rightharpoonup 0$ weakly, thus it also follows that $R_m \to \infty$ as $m \to \infty$, and $\tilde{\Omega}_\infty$ either equals $\mathbb{R}^n$ or is a half-space. Now we distinguish two cases:
(1°) $R_m \operatorname{dist}(x_m, \partial\Omega) \le c < \infty$, uniformly, in which case $\tilde{\Omega}_\infty$ is a half-space, or
(2°) $R_m \operatorname{dist}(x_m, \partial\Omega) \to \infty$, in which case $\tilde{\Omega}_m \to \tilde{\Omega}_\infty = \mathbb{R}^n$.

Since in each case for any $\varphi \in C_0^\infty(\tilde{\Omega}_\infty)$ we have that $\varphi \in C_0^\infty(\tilde{\Omega}_m)$ for large m, there holds

$$\langle \varphi, DE_0(v^0; \tilde{\Omega}_\infty)\rangle = \lim_{m\to\infty} \langle \varphi, DE_0(\tilde{v}_m; \tilde{\Omega}_m)\rangle \le C \lim_{m\to\infty} ||DE_0(v_m)||_{H^{-1}} = 0\,,$$

for all such φ, and $v^0 \in H_0^{1,2}(\tilde{\Omega}_\infty)$ is a weak solution of (3.1) on $\tilde{\Omega}_\infty$. But if $\tilde{\Omega}_\infty = \mathbb{R}^n_+$, by Theorem 1.3 then v^0 must vanish identically. Thus (1°) is impossible, and we are left with (2°). Thus we may now also choose a sequence $(\overline{R}_m)$ such that $\tilde{R}_m := R_m(\overline{R}_m)^{-1} \to \infty$ while $\overline{R}_m \operatorname{dist}(x_m, \partial\Omega) \to \infty$ as $m \to \infty$.

To conclude the proof, again let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function satisfying $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_1(0)$, $\varphi \equiv 0$ outside $B_2(0)$, and let

$$w_m(x) = v_m(x) - R_m^{\frac{n-2}{2}} v^0\big(R_m(x - x_m)\big) \cdot \varphi\big(\overline{R}_m(x - x_m)\big) \in H_0^{1,2}(\Omega).$$

That is,

$$\tilde{w}_m(x) = R_m^{\frac{2-n}{2}} w_m(x/R_m + x_m) = \tilde{v}_m(x) - v^0(x)\varphi(x/\tilde{R}_m)\ .$$

Set $\varphi_m(x) = \varphi\big(x/\tilde{R}_m\big)$. Note that

$$\int_{\mathbb{R}^n} \big|\nabla\big(v^0(\varphi_m - 1)\big)\big|^2 \, dx$$
$$\le C \int_{\mathbb{R}^n} |\nabla v^0|^2 (\varphi_m - 1)^2 \, dx + C \int_{\mathbb{R}^n} |v^0|^2 \big|\nabla(\varphi_m - 1)\big|^2 \, dx$$
$$\le C \int_{\mathbb{R}^n \setminus B_{\tilde{R}_m}(0)} |\nabla v^0|^2 \, dx + C\tilde{R}_m^{-2} \int_{B_{2\tilde{R}_m}(0)\setminus B_{\tilde{R}_m}(0)} |v^0|^2 \, dx\ .$$

But $\nabla v^0 \in L^2(\mathbb{R}^n)$. Therefore the first term tends to 0 as $m \to \infty$, while by Hölder's inequality also the second term

$$\tilde{R}_m^{-2} \int_{B_{2\tilde{R}_m}(0)\setminus B_{\tilde{R}_m}(0)} |v^0|^2 \, dx \le C \left(\int_{B_{2\tilde{R}_m}(0)\setminus B_{\tilde{R}_m}(0)} |v^0|^{2^*} \, dx \right)^{2/2^*} \to 0$$

as $m \to \infty$. Thus we have $\tilde{w}_m = \tilde{v}_m - v^0 + o(1)$, where $o(1) \to 0$ in $D^{1,2}(\mathbb{R}^n)$. As in the proof of Lemma 2.3 we then obtain the expansion

$$E_0(w_m) = E_0(\tilde{w}_m) = E_0(\tilde{v}_m) - E_0(v^0) + o(1) = E_0(v_m) - E_0(v^0) + o(1).$$

Moreover, by a similar reasoning for any sequence of testing functions $\psi_m \in H^{1,2}(\tilde{\Omega}_m)$ with $\|\psi_m\|_{H^{1,2}} \le 1$ with error $o(1) \to 0$ as $m \to \infty$ we obtain

$$\begin{aligned}
\langle \psi_m, DE_0(\tilde{w}_m; \tilde{\Omega}_m)\rangle &= \int_{\mathbb{R}^n} \left(\nabla \tilde{w}_m \nabla \psi_m - \tilde{w}_m |\tilde{w}_m|^{2^*-2} \psi_m \right) dx \\
&= \int_{\mathbb{R}^n} \left(\nabla \tilde{v}_m \nabla \psi_m - \tilde{v}_m |\tilde{v}_m|^{2^*-2} \psi_m \right) dx \\
&\qquad - \int_{\mathbb{R}^n} \left(\nabla v^0 \nabla \psi_m - v^0 |v^0|^{2^*-2} \psi_m \right) dx + o(1) \\
&= \langle \psi_m, DE_0(\tilde{v}_m; \tilde{\Omega}_m)\rangle - \langle \psi_m, DE_0(v^0; \mathbb{R}^n)\rangle + o(1).
\end{aligned}$$

Thus, we find

$$\begin{aligned}
&\|DE_0(w_m; \Omega)\|_{H^{-1}} \\
&\quad = \|DE_0(\tilde{w}_m; \tilde{\Omega}_m)\|_{H^{-1}} = \sup_{\substack{\psi \in H^{1,2}(\tilde{\Omega}_m) \\ \|\psi\|_{H^{1,2}} \le 1}} \langle \psi, DE_0(\tilde{w}_m; \tilde{\Omega}_m)\rangle \\
&\quad \le \|DE_0(\tilde{v}_m; \tilde{\Omega}_m)\|_{H^{-1}} + \|DE_0(v^0; \mathbb{R}^n)\|_{H^{-1}} + o(1) \\
&\quad = \|DE_0(v_m; \Omega)\|_{H^{-1}} + o(1) \ \to 0 \qquad (m \to \infty) \, .
\end{aligned}$$

This concludes the proof. □

Positive Solutions on Annular-shaped Regions

With the aid of Theorem 3.1 we can now show the existence of positive solutions to (1.1), (1.3) on perturbed annular domains for $\lambda = 0$.

The following result is due to Coron [2]:

3.4 Theorem. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ satisfying the following condition: There exist constants $0 < R_1 < R_2 < \infty$ such that*

(1°) $\qquad \Omega \supset \{x \in \mathbb{R}^n \ ; \ R_1 < |x| < R_2\} \, ,$

(2°) $\qquad \overline{\Omega} \not\supset \{x \in \mathbb{R}^n \ ; \ |x| < R_1\} \, .$

Then, if R_2/R_1 is sufficiently large, problem (1.1), (1.3) for $\lambda = 0$ admits a positive solution $u \in H_0^{1,2}(\Omega)$.

Again note that the solution u must have an energy above the compactness threshold given by Lemma 2.3.

The idea of the proof is to argue by contradiction and to use a minimax method for $S = S_0(\,\cdot\,;\Omega)$. Recall that the Sobolev constant in $\mathbb{R}^n$ is attained at the functions u_ε^* defined in (2.5). Consider the sphere $\Sigma = \{x \in \mathbb{R}^n; |x|^2 = R_1R_2\}$ in the annulus $\{x \in \mathbb{R}^n; R_1 < |x| < R_2\}$. For suitably small $\varepsilon > 0$ then consider the collection $A = \{u_\varepsilon^*(\cdot - \sigma)\varphi \in H_0^{1,2}(\Omega);\ \sigma \in \Sigma\}$ of shifted functions u_ε^*, truncated with a suitable cut-off function $\varphi \in C_0^\infty(\Omega)$. If we regard the center of mass, as explained below in detail, then for sufficiently small $\varepsilon > 0$ the set A is homeomorphic to the sphere Σ, which by assumption (2°) is not contractible in Ω. After normalizing, $A \subset M = \{u \in H_0^{1,2}(\Omega) :\ \|u\|_{L^{2^*}} = 1\}$.

On the other hand, any set A of positive functions in M is contractible in M. Moreover, for a sufficiently large ratio R_2/R_1 we can contract A in such a way that the corresponding minimax-value β is smaller than the number $2^{2/n}S$. If we assume that (1.1), (1.3) does not admit a positive solution, by Theorem 3.1 and Remark 3.2 this is the smallest number above S where the Palais-Smale condition fails in M. By applying the deformation lemma Theorem II.3.11 we then conclude that the set A can be contracted below the level $S + \delta$ for any $\delta > 0$. For suitably small $\delta > 0$, finally, such a contraction will induce a contraction of Σ in Ω, and the desired contradiction will result.

Proof. We may assume $R_1 = (4R)^{-1} < 1 < 4R = R_2$. Consider the unit sphere

$$\Sigma = \{x \in \mathbb{R}^n\ ;\ |x| = 1\}\ .$$

For $\sigma \in \Sigma,\ x \in \mathbb{R}^n,\ 0 \le t < 1$ let

$$u_t^\sigma(x) = \left[\frac{1-t}{(1-t)^2 + |x - t\sigma|^2}\right]^{\frac{n-2}{2}} \in D^{1,2}(\mathbb{R}^n)\ .$$

Note that S is attained on any such function u_t^σ, and u_t^σ "concentrates" at σ as $t \to 1$. Moreover, letting $t \to 0$ we have

$$u_t^\sigma \to u_0 = \left[\frac{1}{1 + |x|^2}\right]^{\frac{n-2}{2}}\ ,$$

for any $\sigma \in \Sigma$. Choose a radially symmetric function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$ on Ω, $\varphi \equiv 1$ on the annulus $\{x\ ;\ \frac{1}{2} < |x| < 2\}$ and $\varphi \equiv 0$ outside the annulus $\{x\ ;\ \frac{1}{4} < |x| < 4\}$. Given any $R \ge 1$, scale

$$\varphi_R(x) = \begin{cases} \varphi(Rx), & 0 \le |x| < R^{-1} \\ 1, & R^{-1} \le |x| < R \\ \varphi(x/R), & R \le |x| \end{cases}$$

and let

$$w_t^\sigma = u_t^\sigma \cdot \varphi_R,\ w_0 = u_0 \cdot \varphi_R \in H_0^{1,2}(\Omega)\ .$$

Note that

$$\int_{\mathbb{R}^n} |\nabla(w_t^\sigma - u_t^\sigma)|^2\ dx \le c \int_{(\mathbb{R}^n \setminus B_{2R}) \cup B_{(2R)^{-1}}} |\nabla u_t^\sigma|^2\ dx$$
$$+ c \cdot R^{-2} \int_{B_{4R} \setminus B_{2R}} |u_t^\sigma|^2\ dx + cR^2 \int_{B_{(2R)^{-1}}} |u_t^\sigma|^2\ dx \to 0$$

as $R \to \infty$, uniformly in $\sigma \in \Sigma,\ t \in [0,1[$. Also consider the normalized functions

$$v_t^\sigma = w_t^\sigma / \|w_t^\sigma\|_{L^{2^*}},\ v_0 = w_0 / \|w_0\|_{L^{2^*}}\ .$$

Since $\|w_t^\sigma - u_t^\sigma\|_{D^{1,2}} \to 0$ as $R \to \infty$ and $\|u_t^\sigma\|_{D^{1,2}} = \|u_0\|_{D^{1,2}} > 0$, we have

$$S(v_t^\sigma; \Omega) = S(w_t^\sigma; \Omega) \to S(u_t^\sigma; \mathbb{R}^n) = S$$

as $R \to \infty$, uniformly in $\sigma \in \Sigma,\ 0 \le t < 1$. In particular, if $R \ge 1$ is sufficiently large, we can achieve that

$$\sup_{\sigma,t} S\big(v_t^\sigma; \Omega\big) < S_1 < 2^{2/n} S$$

for some constant $S_1 \in \mathbb{R}$.

Fix such an R and suppose that Ω satisfies (1°), (2°) of the Theorem with $R_2 = 4R = R_1^{-1}$ for this number R. Let M be the set

$$M = \{u \in H_0^{1,2}(\Omega)\ ;\ \int_\Omega |u|^{2^*}\ dx = 1\}\ ,$$

and for $u \in M$ let

$$F(u) = \int_\Omega x |\nabla u|^2\ dx$$

denote its center of mass. Suppose (1.1), (1.3) does not admit a positive solution. This is equivalent to the assertion that

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2\ dx - \frac{1}{2^*} \int_\Omega |u|^{2^*}\ dx$$

does not admit a critical point $u > 0$. By Remark 3.2, therefore, E satisfies (P.-S.)$_\beta$ on $H_0^{1,2}(\Omega)$ for $\frac{1}{n} S^{n/2} < \beta < \frac{2}{n} S^{n/2}$. Equivalently, $S_0(\,\cdot\,; \Omega)$ satisfies (P.-S.)$_\beta$ on M for $S < \beta < 2^{2/n} S$. Moreover, $S_0(\,\cdot\,; \Omega)$ does not admit a critical value in this range.

By the deformation lemma Theorem II.3.11 and Remark II.3.12, therefore, for any β in this range there exists $\varepsilon > 0$ and a flow $\Phi\colon M \times [0,1] \to M$ such that

$$\Phi(M_{\beta+\varepsilon}, 1) \subset M_{\beta-\varepsilon}\ ,$$

where

$$M_\beta = \{u \in M\ ;\ S(u; \Omega) < \beta\}\ .$$

Given $\delta > 0$, we may cover the interval $[S + \delta, S_1]$ by finitely many such ε-intervals and compose the corresponding deformations to obtain a flow $\Phi\colon M \times [0,1] \to M$ such that

$$\Phi(M_{S_1}, 1) \subset M_{S+\delta} \ .$$

Moreover, we may assume that $\Phi(u,t) = u$ for all u with $S(u;\Omega) \leq S + \delta/2$.

On the other hand, it easily follows either from Theorem I.4.8 or Theorem 3.1 that, given any neighborhood U of $\overline{\Omega}$, there exists $\delta > 0$ such that $F(M_{S+\delta}) \subset U$. Indeed, for any sequence (u_m), where $u_m \in M_{S+\frac{1}{m}}$, by Theorem I.4.8 and Theorem 1.2 there exists a subsequence such that, as $m \to \infty$,

$$|u_m|^{2^*} \, dx \rightharpoonup \delta_{x^{(0)}} \ , \ |\nabla u_m|^2 \, dx \rightharpoonup S\delta_{x^{(0)}}$$

for some $x^{(0)} \in \bar{\Omega}$. Since Ω is smooth we may choose a neighborhood U of $\overline{\Omega}$ such that any point $p \in U$ has a unique nearest neighbor $q = \pi(p) \in \overline{\Omega}$ and such that the projection π is continuous. Let $\delta > 0$ be determined for such a neighborhood U, and let $\Phi\colon M \times [0,1] \to M$ be the corresponding flow constructed above. The map $h\colon \Sigma \times [0,1] \to \overline{\Omega}$, given by

$$h(\sigma, t) = \pi\Big(F\big(\Phi(v_t^\sigma, 1)\big)\Big) \ ,$$

then is well-defined, continuous, and satisfies

$$h(\sigma, 0) = \pi\Big(F\big(\Phi(v_0, 1)\big)\Big) =: x_0 \in \overline{\Omega} \ , \qquad \text{for all } \sigma \in \Sigma$$

$$h(\sigma, 1) = \sigma, \qquad \text{for all } \sigma \in \Sigma \ .$$

Hence h is a contraction of Σ in $\overline{\Omega}$, contradicting (2°). □

Actually, the effect of topology is much stronger than indicated by Theorem 3.4. In a penetrating analysis, Bahri-Coron [1] have obtained the following result; see also Bahri [2]:

3.5 Theorem. *Suppose Ω is a domain in $\mathbb{R}^n$ such that*

$$H_d(\Omega, \mathbb{Z}_2) \neq 0$$

for some $d > 0$. Then (1.1), (1.3) admits a positive solution for $\lambda = 0$.

Note that if $\Omega \subset \mathbb{R}^3$ is non-contractible then either $H_1(\Omega, \mathbb{Z}_2)$ or $H_2(\Omega, \mathbb{Z}_2) \neq 0$ and the conclusion of Theorem 3.5 holds. It is conjectured that a similar result will also hold for $n \geq 4$.

3.6 Notes. (1°) Benci-Cerami-Passaseo [1] – see, in particular, Theorem 3.1 – have further refined the "method of photography" used in the proof of Theorem 3.4 above and have applied it to obtain multiple solutions of semilinear elliptic equations also on unbounded domains.

(2°) Schoen-Zhang [1] have obtained pointwise estimates for concentrating solutions $u_m > 0$ with $u_m \rightharpoonup u_0$ weakly in $H^{1,2}(S^3)$ to subcritical equations

$$-\Delta u_m + \frac{n(n-2)}{4} u_m = K u_m^{p_m - 1} \text{ on } S^3,$$

comparing u_m to a "bubble" $R_m^{\frac{n-2}{2}} u_1^*(R_m(x - x_m))$ of the "right" scale R_m, centered around a suitable point x_m. Similar and improved pointwise estimates for bubbling families of solutions $u_m > 0$ to

$$-\Delta u_m + h_m u_m = u_m^{2^*-1}$$

on manifolds have been obtained by Druet-Hebey-Robert [1] in the case when $h_m \to h_0$ in $C^{0,\alpha}$ while $u_m \rightharpoonup u_0$ weakly in $H^{1,2}$.
(3°) Results similar to Theorem 3.1 have been established for higher order equations of Yamabe type by Hebey-Robert [1].
(4°) The work of Brezis-Merle [1] Li-Shafrir [1], Adimurthi-Struwe [1] and, finally, Druet [1] shows that similar geometric quantization patterns also characterize the blow-up behavior for concentrating solutions to semilinear elliptic equations of critical exponential growth on a planar domain. In recent work of Struwe [28] the results of Druet [1] have been extended to critical semilinear elliptic equations of fourth order on four-dimensional domains.

4. The Yamabe Problem

Equation (1.1) arises in a geometric context in the problem whether a given metric g_0 on a closed manifold M of dimension $n \geq 3$ with scalar curvature $R_{g_0} = R_0$ can be deformed conformally to a metric g of constant scalar curvature. If we let

$$g = u^{\frac{4}{n-2}} g_0 \ ,$$

where $u > 0$ gives the conformal factor, the scalar curvature $R = R_g$ of the metric g is given by the equation

$$-\frac{4(n-1)}{n-2} \Delta_0 u + R_0 u = R u^{2^*-1}, \tag{4.1}$$

where $\Delta_0 = \Delta_{g_0}$ is the Laplace-Beltrami operator on the manifold M with respect to the original metric g_0; see Yamabe [1], T. Aubin [2], [3]. Observe that by its intrinsic geometric meaning Equation (4.1) is conformally invariant; that is, if u solves (4.1) on (M, g_0) and if

$$g_0 = v^{\frac{4}{n-2}} \tilde{g}_0, \qquad v > 0$$

is conformal to a metric $\tilde{g}_0$ on M, then $\tilde{u} = uv$ satisfies (4.1) on $(M, \tilde{g}_0)$ with R_0 replaced by the scalar curvature $\tilde{R}_0$ of the metric $\tilde{g}_0$; see also Aubin [3; Proposition, p. 126].

4.1 The sphere. Of particular interest is the case of the sphere $M = S^n$ with the standard metric induced by the embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. We center S^n at the origin of $\mathbb{R}^n \times \mathbb{R}$. Via the inverse of stereographic projection

$$\mathbb{R}^n \ni x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right) \in S^n \subset \mathbb{R}^{n+1} \ ,$$

S^n induces a metric $g = \frac{4}{[1+|x|^2]^2} g_{\mathbb{R}^n}$ of constant scalar curvature $R_g = n(n-1)$ on $\mathbb{R}^n$, conformal to the Euclidean metric. The classical solution

$$u(x) = \frac{[n(n-2)]^{\frac{n-2}{4}}}{[1+|x|^2]^{\frac{n-2}{2}}}$$

of (3.1) on $\mathbb{R}^n$ thus reappears in the conformal factor that changes the flat Euclidean metric to a metric of constant scalar curvature, and the invariance of (3.1) under scaling

$$u \mapsto u_{r,x_0}(x) = r^{\frac{n-2}{2}} u\big(r(x-x_0)\big)$$

reflects the action of the group of conformal diffeomorphism of S^n via translations and dilations in the chart $\mathbb{R}^n$.

The Variational Approach

We can give a variational formulation of (4.1) analogous to the one given for (1.1)–(1.3). Let $H^{1,2}(M, g_0)$ be the set of Sobolev functions $u\colon M \to \mathbb{R}$ such that in local coordinates on M, with $g_0^{ij}(x)$ denoting the coefficients of the inverse of the metric $g_0(x)$, and with volume element

$$d\mu_0 = d\mu_{g_0} = \sqrt{\det(g_0)}\, dx,$$

we have

$$\|u\|^2_{H^{1,2}} := \int_M \big(|\nabla u|_0^2 + |u|^2\big) d\mu_0 < \infty$$

where $|\nabla u|_0^2 = g_0^{ij} \partial_i u \partial_j u$. By convention, we tacitly sum over repeated indices.

Let $c_n = \frac{4(n-1)}{n-2}$. In analogy with the Sobolev ratio in the Euclidean case, for $0 \neq u \in H^{1,2}(M, g_0)$ we define

$$S(u) = S(u; (M, g_0)) = \frac{\int_M \big(c_n |\nabla u|_0^2 + R_0 |u|^2\big) d\mu_0}{\big(\int_M |u|^{2^*} d\mu_0\big)^{2/2^*}} \ .$$

Then S is of class C^1 on the space $H^{1,2}(M, g_0) \setminus \{0\}$, and critical points $u > 0$ of S on the unit sphere in $L^{2^*}(M, g_0)$ correspond to solutions of (4.1) with constant scalar curvature $R = S(u)$, the total scalar curvature of the metric $g = u^{\frac{4}{n-2}} g_0$. In particular, if the Yamabe constant

$$Y_0 = Y(M, g_0) = \inf \big\{ S(u);\, 0 \neq u \in H^{1,2}(M, g_0) \big\} \ ,$$

on (M, g_0) is attained at a function $u > 0$, Equation (4.1) will hold with $R \equiv Y_0$.

If $Y_0 \neq 0$, a constant multiple of u then will solve (4.1) with $R = 1$ or $R = -1$. Thus, depending on the sign of $Y(M, g_0)$, the latter question can also be approached as a minimax problem for the free functional

$$E(u) = E(u; (M, g_0)) = \frac{1}{2} \int_M \big(c_n |\nabla u|_0^2 + R_0 |u|^2\big) d\mu_0 \pm \frac{1}{2^*} \int_M |u|^{2^*} d\mu_0,$$

where the "+"-sign ("−"-sign) is valid if $Y(M, g_0)$ is negative (positive). If $Y(M, g_0) = 0$, also $R = 0$ and (4.1) reduces to a linear equation.

Note that the Yamabe constant $Y(M, g_0)$ is independent of the metric g_0 representing a conformal class. In particular, we have $n(n-1) = Y(S^n, g_{S^n}) = Y(\mathbb{R}^n, g_{\mathbb{R}^n}) = c_n S$, where S is the Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$. An argument as in Remark I.4.5 then also shows that there holds $Y(M, g_0) \leq Y(S^n, g_{S^n})$ for any (M, g_0).

If $Y(M, g_0) < 0$, the functional E is coercive and weakly lower semicontinuous on $H^{1,2}(M, g_0)$, as may easily be verified by the reader. Hence in this case the existence of a non-negative critical point follows from the direct methods; see Chapter I. Alternatively, one may employ the method of sub- and supersolutions to (4.1) used by Kazdan-Warner [1] or Loewner-Nirenberg [1]; see also Sections I.2.3–I.2.6.

The difficult case is the case when $Y(M, g_0) > 0$. Years after Yamabe's [1] first – unsuccessful – attempt to solve (4.1) for general manifolds (M, g_0), Trudinger [1; Theorem 2, p. 269] obtained a rigorous existence result for small positive $Y(M, g_0)$. His approach then was refined by Aubin [2]. By using optimal Sobolev estimates, Aubin was able to show the following result on which Lemma 2.2 above was modeled.

4.2 Lemma. *If $Y(M, g_0) < Y(S^n, g_{S^n})$, then $Y(M, g_0)$ is attained at a solution $u > 0$ to (4.1), inducing a conformal metric of constant scalar curvature.*

As a companion result in the spirit of Lemma 2.3 we find that E satisfies (P.-S.)$_\beta$ on $H^{1,2}(M, g_0)$ for $\beta < \frac{1}{n} Y(S^n, g_{S^n})^{n/2}$.

Aubin succeeded in showing that the condition $Y(M, g_0) < Y(S^n, g_{S^n})$ is satisfied if (M, g_0) is of dimension $n \geq 6$ and not locally conformally flat. By means of the "positive mass theorem" Schoen [1], finally, was able to show that this condition also holds in all the remaining cases when $(M, g_0) \neq (S^n, g_{S^n})$. Thus, we obtain the following result.

4.3 Theorem. *On any smooth, compact Riemannian manifold (M, g_0) without boundary of dimension $n \geq 3$ there exists a conformal metric of constant scalar curvature, a "Yamabe metric".*

The Locally Conformally Flat Case

Particularly powerful tools are available for dealing with the Yamabe problem on a locally conformally flat manifold (M, g_0). In fact, a complete Morse theory for the Yamabe problem may be obtained in this case.

4.4 The developing map. A manifold (M, g_0) is locally conformally flat if for every point $p \in M$ there is a neighborhood U of p in M and a conformal diffeomorphism Φ from (U, g_0) to S^n, endowed with the standard metric. Locally conformally flat manifolds were studied by Kuiper and, in particular, by Schoen-Yau [1], whom we follow in the sequel. If $n \geq 3$ and if M is simply connected, by a standard monodromy argument the local diffeomorphisms $\Phi: U \subset M \to S^n$ can be extended to a conformal immersion $\Phi: M \to S^n$, the developing map, which is unique up to composition with a conformal, or Möbius, transformation of S^n. Similarly, for an arbitrary locally conformally flat n-manifold (M, g_0), $n \geq 3$, there is a developing map Φ from the universal cover $\tilde{M}$ of M to S^n, and the fundamental group $\pi_1(M)$ acting on $\tilde{M}$ is mapped into the Möbius group by a homomorphism called the holonomy representation.

One of the main accomplishments in Schoen-Yau [1] is to exhibit a large class of locally conformally flat manifolds where the developing maps are injective.

4.5 The conformal Laplace operator. The left-hand side of (4.1) defines the conformal Laplace operator on (M, g_0),

$$L_0 = L_{g_0} = -c_n \Delta_0 + R_0 \ ,$$

scaled with the factor $c_n = \frac{4(n-1)}{n-2}$ for convenience. Recall that throughout this section we only consider closed manifolds (M, g_0), that is, compact manifolds without boundary. Let λ_0 be the lowest eigenvalue of L_0 and $u_0 > 0$ a corresponding eigenfunction. By (4.1), the metric $g = u_0^{4/(n-2)} g_0$ has a scalar curvature R_g of constant sign, in fact, of the same sign as λ_0, which is the same as the sign of the Yamabe invariant $Y(M, g_0)$. Thus, according to the sign of $Y(M, g_0)$, a closed locally conformally flat manifold (M, g_0) may be classified conformally invariantly as scalar positive, scalar negative, or scalar flat.

In the following we may restrict ourselves to the case that (M, g_0) is scalar positive as the most interesting case. For such manifolds we have the following result of Schoen-Yau [1], Proposition 3.3 and 4.4.

4.6 Theorem. *Suppose (M, g_0) is closed, locally conformally flat and scalar positive. Then the developing map $\Phi: \tilde{M} \to S^n$ is injective and $\Gamma = \partial(\Phi(\tilde{M}))$ has vanishing Newtonian capacity.*

Via the covering map $\pi: \tilde{M} \to M$ we lift g_0 to an equivariant metric $\tilde{g}_0 = \pi^* g_0$. Using Theorem 4.6, we can push this metric forward conformally to a complete metric

$$h_0 = \Phi_* \tilde{g}_0 = v_0^{\frac{4}{n-2}} g_{S^n}$$

on $\Phi(\tilde{M}) \subset S^n$, conformal to the standard metric g_{S^n} on S^n.

By a second result of Schoen-Yau [1], Proposition 2.6, the conformal factor v_0 diverges uniformly near the boundary Γ of $\Phi(\tilde{M})$.

4.7 Proposition. *There exists a constant $c > 0$ such that for $x \in \Phi(\tilde{M})$ there holds $v_0(x) \geq c\,\mathrm{dist}(x, \Gamma)^{\frac{2-n}{2}}$. The constant $c = c(M, g_0)$ depends only on the conformal class of g_0 and bounds for the curvature of g_0 and its derivatives with respect to g_0.*

As a final normalization, upon replacing g_0 by a constant multiple of g_0, if necessary, we may assume that (M, g_0) has unit volume. Similarly, we will impose the volume constraint

$$Vol(g) = \int_M d\mu = \int_M u^{2^*} d\mu_0 = 1. \tag{4.2}$$

on comparison metrics $g = u^{\frac{4}{n-2}} g_0$, where $d\mu = d\mu_g = u^{2^*} d\mu_0$.

The Yamabe Flow

With these prerequisites we now turn to the construction of a Yamabe metric g_∞ on (M, g_0). Instead of using one of the standard variational methods presented so far, following ideas of Hamilton [1] we will approach the problem by means of the parabolic evolution equation corresponding to the "L^2-gradient" flow for the Yamabe energy $S(u)$, whose convergence on a locally conformally flat scalar positive manifold was demonstrated by Ye [1].

4.8 The evolution problem. The metric g_∞ is determined as the limit as $t \to \infty$ of the "Yamabe flow" of metrics $(g(t))_{t \geq 0}$ issuing from $g(0) = g_0$. Letting

$$g(t) = u(\cdot, t)^{\frac{4}{n-2}} g_0 \tag{4.3}$$

with $R_{g(t)} = R(t)$, this flow is defined by letting $u\colon M \times [0, \infty[\to \mathbb{R}$ solve the evolution problem

$$\frac{\partial u^p}{\partial t} = s u^p - L_0 u = (s - R) u^p \tag{4.4}$$

for $p = 2^* - 1 = \frac{n+2}{n-2}$ with initial condition

$$u(\cdot, 0) = 1$$

and with a function $s = s(t)$ determined in such a way that the volume constraint (4.2) is preserved. From the equation

$$0 = \frac{p}{2^*}\frac{d}{dt}Vol(g(t)) = \int_M u\frac{\partial u^p}{\partial t}\, d\mu_0 = \int_M u(-L_0u + su^p)\, d\mu_0$$
$$= -\int_M \big(c_n|\nabla u|_0^2 + R_0u^2\big)d\mu_0 + s\int_M u^{2^*}\, d\mu_0$$

we then deduce that

$$s(t) = S(u(t)) \quad \text{for all } t. \tag{4.5}$$

Multiplying (4.4) by u_t and integrating over M, in view of (4.2) we also obtain the energy identity

$$2p\int_M u^{p-1}|u_t|^2\, d\mu_0 + \frac{d}{dt}S(u(t)) = \frac{2s}{2^*}\frac{d}{dt}Vol(g(t)) = 0\ .$$

Hence, the function $t \mapsto s(t)$ is non-increasing with a well-defined limit

$$s_\infty = \lim_{t\to\infty} s(t) \geq Y(M, g_0) > 0.$$

In addition, we have the a-priori estimate

$$2p\int_0^\infty \int_M u^{p-1}|u_t|^2\, d\mu_0 dt = s(0) - s_\infty \leq S(1) - Y(M, g_0) < \infty$$

for any smooth global solution u of (4.4), (4.5). In particular, if u together with its space-time derivatives is uniformly bounded on $M \times [0, \infty[$, a sequence $u(t)$ as $t \to \infty$ will accumulate at a time-independent solution u_∞ of (4.4), that is, a solution of (4.1) with $R = s_\infty$.

When (M, g_0) is locally conformally flat the necessary a-priori estimates are a consequence of the following logarithmic gradient bound for equation (4.4), due to Ye [1], Theorem 4.

4.9 Theorem. *Suppose that (M, g_0) is a closed, locally conformally flat n-manifold, $n \geq 3$. There exists a constant $C = C(M, g_0)$ such that for any smooth solution $u\colon M \times [0, T[\to \mathbb{R}$ of (4.4), (4.5) with $u(\cdot, 0) = 1$ there holds*

$$\sup_{M\times[0,T[} \frac{|\nabla_{g_0} u|}{u} \leq C\ .$$

By integrating along a shortest geodesic connecting points on M where $u(t)$ achieves its maximum, respectively, its minimum, the estimate of Theorem 4.9 implies the uniform Harnack inequality

$$\inf_M u(t) \geq c \sup_M u(t)$$

for $0 < t < T$ with a uniform constant $c > 0$. Hence, in view of the volume constraint (4.2), the solution u is uniformly bounded from above and away from 0. But then Equation (4.4) is uniformly parabolic, and we obtain uniform

derivative estimates for u in terms of the data, as well. In particular, any local solution u may be extended for all time and, as $t \to \infty$ suitably, by our remarks above will converge smoothly to a limit u_∞ inducing a conformal metric g_∞ of constant scalar curvature on M. Theorem 4.9 thus implies the existence of a Yamabe metric on any closed, locally conformally flat manifold (M, g_0) of dimension $n \geq 3$ and completes the proof of Theorem 4.3 in this case.

The Proof of Theorem 4.9 (following Ye [1])

We only consider the scalar positive case $Y(M, g_0) > 0$. In this case by Theorem 4.6 we can transplant the problem from our abstract manifold M to the sphere via the developing map Φ defined above. We then consider the evolution of the metrics

$$h(t) = \Phi_* \tilde{g}(t) = \tilde{u}(t)^{\frac{4}{n-2}} h_0 = v(t)^{\frac{4}{n-2}} g_{S^n}$$

on $\Phi(\tilde{M}) \subset S^n$, where $\tilde{g}(t) = \pi^* g(t)$, $\tilde{u}(y, t) = u(\pi(\Phi^{-1}(y)), t)$, and where $v(t) = \tilde{u}(t) v_0$. Let L_{h_0} denote the conformal Laplace operator on $(\Phi(\tilde{M}), h_0)$.

Since $\pi \circ \Phi^{-1}$ defines a local isometry between $(\Phi(\tilde{M}), h_0)$ and (M, g_0), from (4.4) we deduce the equation

$$\frac{\partial \tilde{u}^p}{\partial t} + L_{h_0} \tilde{u} = s \tilde{u}^p \tag{4.6}$$

on $\Phi(\tilde{M}) \times [0, T[$. But (4.1) implies

$$\tilde{u}^{-p} L_{h_0} \tilde{u} = R_{h(t)} = v^{-p} L_{S^n} v \ .$$

Hence, in terms of the round spherical metric g_{S^n} as background metric on $\Phi(\tilde{M}) \subset S^n$ we can also express (4.4) as the flow equation

$$\frac{\partial v^p}{\partial t} + L_{S^n} v = s v^p \ . \tag{4.7}$$

Moreover, the condition $\tilde{u}(\cdot, 0) = u(\cdot, 0) = 1$ yields the initial condition

$$v \mid_{t=0} = v_0 \ . \tag{4.8}$$

Fix a fundamental domain $N \subset \Phi(\tilde{M})$ for the action of the Kleinian group G corresponding to $\pi_1(M)$ via the holonomy representation. For the proof of Theorem 4.9 it suffices to establish a bound for

$$\sup_{N \times [0, T[} \frac{|\nabla_{g_{S^n}} v|}{v} \ . \tag{4.9}$$

We may assume that there is an open neighborhood V of the closure of N such that $\operatorname{dist}(V, \Gamma) > 0$, where $\Gamma = \partial\big(\Phi(\tilde{M})\big)$.

Given $q_0 \in N$, we introduce conformal charts for a neighborhood of q_0 and for $S^n \setminus \{q_0\}$, as follows. After a rotation, we may assume that q_0 is the

north pole $q_0 = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Then let $F\colon S^n \setminus \{q_0\} \to \mathbb{R}^n$ denote stereographic projection from q_0 with inverse

$$F^{-1}(x) = \left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1} \right) , \ x \in \mathbb{R}^n .$$

Via stereographic projection from the south pole $-q_0$, similarly we obtain the coordinate representation

$$G(x) = \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right)$$

of $S^n \setminus \{-q_0\}$. Note that $G(0) = q_0$ and that we have the relation $F^{-1}(x) = G(\frac{x}{|x|^2})$ for $x \neq 0$. We pull back the metric $h(t) = v(t)^{\frac{4}{n-2}} g_{S^n}$ via F^{-1} to obtain a conformal metric on $\mathbb{R}^n$, given by

$$(F^{-1})^* h(t) = w(t)^{\frac{4}{n-2}} g_{\mathbb{R}^n} .$$

Since

$$(F^{-1})^* g_{S^n} = \left(\frac{2}{1+|x|^2} \right)^2 g_{\mathbb{R}^n} ,$$

it follows that

$$w(x,t) = \left(\frac{2}{1+|x|^2} \right)^{\frac{n-2}{2}} v\Big(G\big(\frac{x}{|x|^2}\big) , t \Big) .$$

Thus, as $|x| \to \infty$ the function $w(t)$ has the asymptotic expansion

(4.10)

$$w(x,t) = \frac{2^{\frac{n-2}{2}}}{|x|^{n-2}} \left(a_0 + \frac{a_i x_i}{|x|^2} + \big(a_{ij} - \frac{n-2}{2} a_0 \delta_{ij}\big) \frac{x_i x_j}{|x|^4} \right) + O\left(\frac{1}{|x|^{n+1}} \right)$$

$$\frac{\partial w}{\partial x_i}(x,t) = \frac{2^{\frac{n-2}{2}}}{|x|^n} \left(a_i - x_i \big((n-2) a_0 + n \frac{a_j x_j}{|x|^2} \big) \right) + O\left(\frac{1}{|x|^{n+1}} \right) ,$$

where

$$a_0 = a_0(t) = v\big(G(0), t\big) > 0, a_i = a_i(t) = \frac{\partial \big(v(\cdot, t) \circ G \big)}{\partial x_i}(0) ,$$

$$a_{ij} = a_{ij}(t) = \frac{1}{2} \frac{\partial^2 \big(v(\cdot, t) \circ G \big)}{\partial x_i \partial x_j}(0) , \ 1 \leq i, j \leq n .$$

The point $y(t)$ with coordinates

$$y_i(t) = \frac{a_i(t)}{(n-2) a_0(t)}$$

is called the *center* of $w(t)$. Note that if we shift coordinates by $y(t)$, then $\tilde{w}(x,t) = w\big(x + y(t), t\big)$ has the asymptotic expansion

$$\begin{aligned}\tilde{w}(x,t) &= \frac{2^{\frac{n-2}{2}}}{|x|^{n-2}}a_0 + O\left(\frac{1}{|x|^n}\right),\\ \frac{\partial\tilde{w}}{\partial x_i}(x,t) &= -\frac{(n-2)2^{\frac{n-2}{2}}}{|x|^n}a_0 x_i + O\left(\frac{1}{|x|^{n+1}}\right)\end{aligned} \tag{4.11}$$

as $|x| \to \infty$.

Moreover, a uniform bound for y gives the desired bound for (4.9). Thus, the proof of Theorem 4.9 will be complete once we establish the following estimate.

4.10 Lemma. *There is a constant C depending only on (M, g_0) and $\operatorname{dist}(V, \Gamma)$, such that $|y(t)| \le C$, uniformly for $q \in N$, $0 \le t < T$.*

Proof. Fix some number $0 < \bar{t} < T$. After a rotation of coordinates and a reflection in the hyperplane $\{x_n = 0\}$, if necessary, we may assume that $y_n(\bar{t}) = \max_i |y_i(\bar{t})|$.

For $\lambda > 0$, $x = (x', x_n) \in \mathbb{R}^n$ denote

$$x^\lambda = (x', 2\lambda - x_n)$$

the image of x after reflection in the hyperplane $\{x_n = \lambda\}$. By the expansion (4.10) and the arguments of Gidas-Ni-Nirenberg [1], Lemma 4.2, there exists $\lambda_0 \ge 1$ such that for any $\lambda \ge \lambda_0$ there holds

$$w(x,0) > w(x^\lambda, 0), \text{ if } x_n < \lambda;\ \frac{\partial w(x,0)}{\partial x_n} < 0, \text{ if } x_n = \lambda\ . \tag{4.12}$$

Note that λ_0 only depends on (M, g_0) and $\operatorname{dist}(V, \Gamma)$. Here and in the following we extend v to all of $S^n \times [0, T\,[$ by letting $v(x,t) = \infty$ for $x \in \Gamma$, as suggested by Proposition 4.7. Similarly, we extend w to $\mathbb{R}^n \times [0, T\,[$ by letting $w(x,t) = \infty$ for $x \in F(\Gamma)$. In particular, by (4.12) the number λ_0 has to be chosen so that $F(\Gamma)$ lies "below" the hyperplane $\{x_n = \lambda_0\}$; in fact, we may choose λ_0 such that $F(\Gamma)$ lies strictly below $\{x_n = \lambda_0\}$.

Since, by assumption, u is smooth on $M \times [0, \bar{t}]$ and hence v is uniformly smooth on $\tilde{V} \times [0, \bar{t}]$, the expansions (4.10) are uniform in $t \in [0, \bar{t}]$. By the same arguments as cited above, therefore, there exists a number $\lambda_1 \ge \lambda_0$ such that for every $\lambda \ge \lambda_1$ and every $t \in [0, \bar{t}]$ there holds

$$w(x,t) > w(x^\lambda, t), \text{ if } x_n < \lambda\ . \tag{4.13}$$

For $\lambda \ge \lambda_0$ define $w^\lambda(x,t) = w(x^\lambda, t)$. We restrict w^λ to the region where $x_n \le \lambda$, $x \notin F(\Gamma)$, $0 \le t \le \bar{t}$. Note that the functions w and w^λ satisfy the evolution equation (4.4) with respect to the flat Euclidean metric on this domain. Moreover, $w = w^\lambda$ for $x_n = \lambda$.

Let

$$I = \{\lambda \in \mathbb{R};\ \lambda > \lambda_0,\ \lambda > \max_{0 \le t \le \bar{t}} y_n(t),\ w^\lambda \le w\}\ .$$

By (4.13) the set I is non-empty. We will show that I is open and (relatively) closed in the interval $]\lambda_0, \infty[$; hence $I =]\lambda_0, \infty[$, which implies that $y_n(\bar{t}) \leq \lambda_0$. Since $\bar{t} < T$ was arbitrary, this then implies the assertion of the lemma.

I is open. Indeed, by (4.12) for $\lambda \geq \lambda_0$ equality $w^\lambda \equiv w$ is impossible. (One might also use the singular set $F(\Gamma)$ to rule out $w^\lambda \equiv w$.) Hence, for any $\lambda \in I$ the parabolic maximum principle and (4.12) imply that we have

$$w^\lambda(x,t) < w(x,t) \text{ for } x_n < \lambda,\ 0 \leq t \leq \bar{t}\ , \tag{4.14}$$

and

$$\frac{\partial w}{\partial x_n}(x,t) < 0 \text{ for } x_n = \lambda,\ 0 \leq t \leq \bar{t}\ . \tag{4.15}$$

Moreover, by Proposition 4.7 and uniform boundedness of w^λ near $F(\Gamma) \times [0, \bar{t}]$, uniformly in $\lambda \geq \lambda_0$, there exists $c > 0$ such that there holds

$$\liminf_{x \to F(\Gamma)} \big(w(x,t) - w^\lambda(x,t)\big) \geq c > 0\ ,$$

uniformly in $t \in [0, \bar{t}]$ and uniformly in $\lambda \geq \lambda_0$.

By (4.10), given any $\Lambda_0 \geq \lambda_0$ there is a constant r_0 such that the estimates (4.14) and (4.15) hold true for $|x| \geq r_0$, for all $t \in [0, \bar{t}]$ and $\lambda \in [\lambda_0, \Lambda_0]$. Thus there exists $\varepsilon > 0$ such that $]\lambda - \varepsilon, \lambda + \varepsilon\ [\subset I$; that is, I is open.

I is also closed. Indeed, suppose that $\lambda > \lambda_0$ belongs to $\bar{I}$. By continuity, we have $w^\lambda \leq w$ and $\lambda \geq \max_{0 \leq t \leq \bar{t}} y_n(t)$. Suppose that $\lambda = \max_{0 \leq t \leq \bar{t}} y_n(t)$. Then $\lambda = y_n(t_0)$ for some $t_0 \in [0, \bar{t}]$. Shift coordinates by $y_n(t_0)$ and denote by $\tilde{w}(x,t) = w(x', \lambda + x_n, t)$, $\tilde{w}^\lambda = w^\lambda(x', \lambda + x_n, t) = w(x', \lambda - x_n, t)$, respectively, the transformed functions. Also let $\tilde{\Gamma}$ be the shifted singular set $F(\Gamma)$. Via F we lift $\tilde{w}, \tilde{w}^\lambda$ back to the sphere by letting

$$\begin{aligned} z^{\frac{4}{n-2}} g_{S^n} &= F^* \tilde{w}^{\frac{4}{n-2}} g_{\mathbb{R}^n}\ , \\ (z^\lambda)^{\frac{4}{n-2}} g_{S^n} &= F^* (\tilde{w}^\lambda)^{\frac{4}{n-2}} g_{\mathbb{R}^n}\ . \end{aligned}$$

Then z, z^λ are defined on $(S^n_- \setminus F^{-1}(\tilde{\Gamma})) \times [0, \bar{t}]$, where S^n_- is the hemisphere corresponding to $x_n < 0$. (z is related to v by a conformal diffeomorphism of S^n fixing the north pole q_0.) Moreover, z and z^λ satisfy (4.7). We also know that $z^\lambda \leq z$ and $z^\lambda = z$ along ∂S^n_-.

Finally, note that by (4.11) the term a_n in the expansion (4.10) for $\tilde{w}(t_0)$ vanishes. In terms of z and z^λ this translates into the condition

$$\frac{\partial z}{\partial \nu}(q_0, t_0) = \frac{\partial z^\lambda}{\partial \nu}(q_0, t_0) = 0\ ,$$

where ν is the outward unit normal on ∂S^n_-. But then the strong parabolic maximum principle implies that $z \equiv z^\lambda$, that is, $w \equiv w^\lambda$. This contradicts (4.12). Hence $\lambda > \max_{0 \leq t \leq \bar{t}} y_n(t)$; that is, $\lambda \in I$, and I is closed.

The proof is complete. □

Convergence of the Yamabe Flow in the General Case

Ye's Theorem 4.9 depends heavily on the special symmetries of the sphere. While it is not yet known if a similar result will hold for the Yamabe flow on an arbitrary closed manifold, it is possible to show convergence of the flow in dimensions $3 \leq n \leq 5$. The following result is due to Schwetlick-Struwe [1] and Brendle [1]. In the case $n = 3$ a different proof was given by Grüneberg [1].

4.11 Theorem. *Let (M, g_0) be a smooth compact, scalar positive manifold without boundary of dimension $3 \leq n \leq 5$. Then there is a unique, global, smooth solution $u > 0$ to the Yamabe flow (4.4) with initial data $u(0) \equiv 1$ such that the associated metrics $g(t) = u(t)^{\frac{4}{n-2}} g_0$ as $t \to \infty$ converge to a metric g_∞ of constant scalar curvature.*

Schwetlick-Struwe [1] obtain the result for initial metrics g_0 with $R_0 = R_{g_0} > 0$ and whose total scalar curvature s_0 is small in the sense that

$$0 < Y_0 \leq s_0 \leq \big(Y(M, g_0)^{n/2} + Y(S^n, g_{S^n})^{n/2}\big)^{2/n} . \tag{4.16}$$

Brendle [1] later was able to remove this restriction. Moreover, he also treated the case when the initial metric g_0 has scalar curvature of varying sign.

The proof of Theorem 4.11 for the general case is very technical. In the following we will therefore focus on the case when $R_{g_0} > 0$ and when g_0 satisfies (4.16). Moreover, we may assume that (M, g_0) is not conformal to (S^n, g_{S^n}).

To simplify the notation, in the following we rescale time with the factor $p = \frac{n+2}{n-2}$ so that our flow equation (4.4) becomes

$$u_t = (s - R)u\,,\ u(0) = 1\,. \tag{4.17}$$

From (4.1) we also obtain the identity

$$R_t = u^{-\frac{n+2}{n-2}} L_0 u_t - \frac{n+2}{n-2} R \frac{u_t}{u}$$

for the evolution of the scalar curvature. Thus, using the invariance property

$$u^{-\frac{n+2}{n-2}} L_0(uv) = Lv = (-c_n \Delta + R)v\,,$$

where $L = L_g$ with $\Delta = \Delta_g$, we may rephrase the former equation as

$$R_t = L(s - R) + \frac{n+2}{n-2} R(R - s) = c_n \Delta R + \frac{4}{n-2} R(R - s)\,. \tag{4.18}$$

Finally, in the scaled time variable the energy inequality reads

$$s_t = \frac{d}{dt}\left(\int_M R\, d\mu\right) = 2\int_M R(s - R)\, d\mu = -2\int_M |R - s|^2\, d\mu \leq 0\,. \tag{4.19}$$

4.12 Global existence. We readily obtain uniform upper and lower a-priori bounds for the solution u to (4.4) on any finite time interval. Standard parabolic estimates then yield global existence of the flow.

First we show that $R > 0$ for all $t \geq 0$. Let $\tilde{R}(t) = e^{\frac{4s_0}{n-2}t} R(t)$. Then, as long as $R \geq 0$, from (4.18), (4.19) we have

$$\tilde{R}_t - c_n \Delta \tilde{R} = \frac{4}{n-2}(R - s + s_0)\tilde{R} \geq \frac{4R}{n-2}\tilde{R} \geq 0.$$

The maximum principle now implies that

$$\tilde{R} \geq \min_M \tilde{R}(0) = \min_M R_0,$$

uniformly, and we conclude that

$$\min_M R(t) \geq \min_M R_0 \cdot e^{-\frac{4s_0}{n-2}t}$$

as claimed.

But then from (4.17) we also obtain the uniform upper bound

$$\sup_M u(t) \leq e^{s_0 t} \sup_M u(0) = e^{s_0 t}$$

for any t. Moreover, we have

4.13 Lemma. *Suppose that $u > 0$ is a smooth function inducing a metric $g = u^{\frac{4}{n-2}} g_0$ with $R \geq 0$, normalized by (4.2). Then with constants C, $q \geq 1$ depending only on (M, g_0) there holds*

$$C \inf_M u \geq ||u||_{L^\infty}^{1-q}.$$

Proof: In view of the equation $L_0 u = R u^{2^*-1} \geq 0$, the weak Harnack inequality implies the estimate

$$C \inf_M u \geq ||u||_{L^{p_0}}$$

with uniform constants C, $p_0 > 0$; see Schwetlick-Struwe [1], Theorem A.2. We may assume $p_0 \leq 2^*$; otherwise, (4.2) gives a uniform lower bound for u and we are done. But then by (4.2) we have

$$1 = ||u||_{L^{2^*}}^{2^*} \leq ||u||_{L^{p_0}}^{p_0} ||u||_{L^\infty}^{2^*-p_0}.$$

Hence we obtain

$$C \inf_M u \geq ||u||_{L^\infty}^{1-2^*/p_0},$$

as desired. □

In view of Lemma 4.13, our uniform upper bound for $u(t)$ then also implies a uniform lower bound for any finite time. Global existence of the flow (4.4) now follows.

4.14 Curvature decay. For $q \geq 1$ also consider the functionals

$$S_q(g) = \int_M R^q \, d\mu \,, \;\; F_q(g) = \int_M |R - s|^q \, d\mu \,,$$

so that $s(g) = S_1(g)$. Note that (4.17) and (4.18) give

$$\begin{aligned}
\partial_t S_q &= q \int_M R^{q-1} R_t \, d\mu + 2^* \int_M R^q \frac{u_t}{u} \, d\mu \\
&= q c_n \int_M R^{q-1} \Delta R \, d\mu + \frac{4q - 2n}{n-2} \int_M R^q (R - s) \, d\mu \\
&= -\frac{4(q-1)c_n}{q} \int_M |\nabla R^{q/2}|^2 \, d\mu + \frac{4q - 2n}{n-2} \int_M (R^q - s^q)(R - s) \, d\mu \,.
\end{aligned}$$

By convexity, moreover, we have

$$(R^q - s^q)(R - s) \geq |R - s|^{q+1} \,.$$

Thus, for $1 \leq q < \frac{n}{2}$ we obtain that

$$\int_0^\infty F_{q+1}(g(t)) \, dt = \int_0^\infty \int_M |R - s|^{q+1} \, d\mu \, dt \leq \frac{n-2}{2n - 4q} S_q(g_0) \,. \tag{4.20}$$

In particular, for $q = 1$ we recover the estimate (4.19). Moreover, for $q < \frac{n}{2}$ we find

$$\liminf_{t \to \infty} F_{q+1}(g(t)) = 0 \,. \tag{4.21}$$

In fact, the convergence is uniform.

4.15 Lemma. *For any $1 \leq p < \frac{n+2}{2}$ there holds $F_p(g(t)) \to 0$ as $t \to \infty$.*

The assertion of Lemma 4.15 even holds true for any $p < \infty$; see Schwetlick-Struwe [1], Lemma 3.3.

Proof of Lemma 4.15. By Hölder's inequality, for $1 \leq q \leq p$ we have $F_q \leq F_p^{q/p}$. It therefore suffices to prove the claim for $p \geq \frac{n+1}{2} \geq 2$. For such p we have

$$\begin{aligned}
\partial_t F_p(g) &= p \int_M (R - s)^{p-1} (R_t - s_t) \, d\mu + 2^* \int_M |R - s|^p \frac{u_t}{u} \, d\mu \\
&= p c_n \int_M (R - s)^{p-1} \Delta R \, d\mu + \frac{4p}{n-2} \int_M |R - s|^p R \, d\mu \\
&\quad - p s_t \int_M (R - s)^{p-1} \, d\mu - 2^* \int_M (R - s)^{p+1} \, d\mu \\
&= -\frac{4(p-1)c_n}{p} \int_M |\nabla (R - s)^{p/2}|^2 \, d\mu + \frac{4p - 2n}{n-2} \int_M (R - s)^{p+1} \, d\mu \\
&\quad + \frac{4ps}{n-2} \int_M |R - s|^p \, d\mu - p s_t \int_M (R - s)^{p-1} \, d\mu \,,
\end{aligned}$$

where $z^\alpha := z|z|^{\alpha-1}$ for all $z \in \mathbb{R}$, any $\alpha > 0$. That is, recalling that $p \geq 2$ and taking account of (4.19), with constants $C = C(p, g_0)$ we have

$$\begin{aligned}\partial_t F_p(g) &\leq -\int_M \big(c_n|\nabla(R-s)^{p/2}|^2 + R|R-s|^p\big)\,d\mu \\ &\quad + 2pF_2(g)F_{p-1}(g) + \big(\frac{4ps_0}{n-2} + s\big)F_p(g) + \big(\frac{|4p-2n|}{n-2} + 1\big)F_{p+1}(g) \\ &\leq -Y_0 F_{p^*}(g)^{\frac{n-2}{n}} + CF_p(g) + CF_{p+1}(g)\,,\end{aligned} \tag{4.22}$$

where $p^* = \frac{np}{n-2}$. Here we also used the fact that our assumption $Y_0 > 0$ implies the uniform Sobolev type inequality

$$Y_0 \left(\int_M |v|^{2^*}\,d\mu\right)^{\frac{n-2}{n}} \leq \int_M \big(c_n|\nabla v|^2 + Rv^2\big)\,d\mu$$

for all $g = g(t)$ and all functions $v \in H^1(M, g)$; moreover, we observe that by Hölder's inequality we can bound $F_2(g)F_{p-1}(g) \leq F_{p+1}(g)$.

Furthermore, by Hölder's inequality again, there holds

$$F_{p+1}(g) = ||R-s||^{p+1}_{L^{p+1}(M,g)} \leq ||R-s||^{p\alpha}_{L^{p^*}(M,g)} ||R-s||^{1+(1-\alpha)p}_{L^p(M,g)},$$

where $\alpha = \frac{n}{2p} < 1$. Applying Young's inequality

$$AB \leq \alpha A^{1/\alpha} + (1-\alpha)B^{1/(1-\alpha)} \leq A^{1/\alpha} + B^{1/(1-\alpha)},$$

for any $\delta > 0$ we then obtain the bound

$$F_{p+1}(g) \leq \delta||R-s||^p_{L^{p^*}(M,g)} + C||R-s||^{p(1+\beta)}_{L^p(M,g)} = \delta F_{p^*}(g)^{\frac{n-2}{n}} + CF_p(g)^{1+\beta}\,,$$

where $C = C(\delta) = \delta^{-\frac{\alpha}{1-\alpha}}$, $\beta = \frac{1}{p(1-\alpha)}$. For sufficiently small $\delta > 0$ from (4.22) we then obtain that

$$\partial_t F_p(g) \leq CF_p(g) + CF_p(g)^{1+\beta}$$

and the claim follows from (4.20), (4.21). □

By means of Lemma 4.15, we can deduce convergence of the flow (4.4) as a consequence of the following proposition.

4.16 Proposition. *For any sequence $t_k \to \infty$ $(k \to \infty)$ there exist constants $0 < \gamma < 1$, C such that for a subsequence there holds*

$$s(t_k) - s_\infty \leq CF_{\frac{2n}{n+2}}(g(t_k))^{\frac{n+2}{2n}(1+\gamma)}\,.$$

Indeed, Proposition 4.16 implies the following result.

4.17 Lemma. *There exist constants $0 < \gamma < 1$, t_0 such that for all $t \geq t_0$ there holds*

$$s(t) - s_\infty \leq F_2(g(t))^{\frac{1+\gamma}{2}} .$$

Proof. We argue indirectly. In view of Lemma 4.15 we thus suppose that for a sequence $t_k \to \infty$ $(k \to \infty)$ we have

$$s(t_k) - s_\infty \geq F_2(g(t_k))^{\frac{1+1/k}{2}} .$$

But by Proposition 4.16 and Hölder's inequality for a subsequence with suitable constants $\gamma > 0$, C there holds

$$s(t_k) - s_\infty \leq C F_{\frac{2n}{n+2}}(g(t_k))^{\frac{n+2}{2n}(1+\gamma)} \leq C F_2(g(t_k))^{\frac{1+\gamma}{2}} .$$

We then conclude that

$$1 \leq C F_2(g(t_k))^{\frac{\gamma-1/k}{2}} ,$$

which for large k contradicts Lemma 4.15. The proof is complete. □

4.18 Lemma. *We have*

$$\int_0^\infty F_2(g(t))^{\frac{1}{2}}\, dt < \infty.$$

Proof. In view of (4.19), the assertion of Lemma 4.17 translates into the differential inequality

$$\frac{d}{dt}(s(t) - s_\infty) = -2F_2(g) \leq -C(s(t) - s_\infty)^{1+\delta}$$

for some $0 < \delta < 1$. It follows that

$$s(t) - s_\infty \leq C t^{-1/\delta}$$

for some constant $C > 0$. Hence from (4.19) we deduce

$$\left(\int_T^{2T} F_2(g(t))^{\frac{1}{2}}\, dt \right)^2 \leq T \int_T^{2T} F_2(g(t))\, dt \leq \frac{T}{2}(s(T) - s(2T)) \leq C T^{1-1/\delta} .$$

By diadic decomposition we conclude

$$\int_1^\infty F_2(g(t))^{\frac{1}{2}}\, dt = \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} F_2(g(t))^{\frac{1}{2}}\, dt \leq C \sum_{k=0}^\infty 2^{-\frac{1-\delta}{2\delta}k} < \infty ,$$

as claimed. □

From Lemma 4.18 weak H^1-convergence of the flow $u(t)$ can be obtained in a straightforward manner. Standard parabolic regularity theory, moreover, yields uniform a-priori bounds for $u(t)$ and its derivatives, which completes the proof of Theorem 4.11; see Brendle [1], Section 3, for details.

It thus remains to give the proof of Proposition 4.16. Fix any sequence $t_k \to \infty$ $(k \to \infty)$ and let $u_k = u(t_k)$. From Lemma 4.15 we conclude that the sequence (u_k) is a Palais-Smale sequence for the Yamabe energy, satisfying the conditions

$$s_k = \int_{M\times\{t_k\}} R\, d\mu = S(u_k) \to s_\infty \geq Y_0$$

and

$$-c_n \Delta_0 u_k + R_0 u_k = s_\infty u_k^{\frac{n+2}{n-2}} + o(1)$$

as $k \to \infty$, with error

$$o(1) = (R_k - s_k) u_k^{\frac{n+2}{n-2}} + (s_k - s_\infty) u_k^{\frac{n+2}{n-2}} \to 0 \text{ in } L^{\frac{2n}{n+2}} \hookrightarrow H^{-1}\,.$$

From the general concentration-compactness result Theorem 3.1 for such sequences we then obtain the following result. Let $r_0 > 0$ denote a lower bound for the injectivity radius on (M, g_0). Fix a function $\varphi \in C_0^\infty(B_{r_0}(0))$ satisfying $\varphi(x) = 1$ on $B_{r_0/2}(0) \subset \mathbb{R}^n$ and for any $x, y \in M$ let $\varphi_y(x) = \varphi(\exp_y^{-1}(x))$, where exp is the exponential map in the metric g_0.

4.19 Lemma. *For any sequence $t_k \to \infty$ $(k \to \infty)$, letting $u_k = u(t_k)$, there exist an integer L and sequences $(x_{k,l})_{k\in\mathbb{N}}$, $(r_{k,l})_{k\in\mathbb{N}}$, $l = 1, \ldots, L$, such that for a subsequence as $k \to \infty$ there holds*

$$u_k(x) - \sum_{l=1}^{L} \varphi_{x_{k,l}}(x)\; r_{k,l}^{\frac{2-n}{2}}\, \bar{u}(r_{k,l}^{-1} \exp_{x_{k,l}}^{-1}(x)) \to u_\infty \text{ in } H^1(M, g_0)\,,$$

where $u_\infty \geq 0$ solves

$$-c_n \Delta_0 u_\infty + R_0 u_\infty = s_\infty u_\infty^{\frac{n+2}{n-2}} \text{ on } M\,, \tag{4.23}$$

and where $\bar{u}$ is a solution to the equation

$$-c_n \Delta_{\mathbb{R}^n} \bar{u} = s_\infty \bar{u}^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n. \tag{4.24}$$

Moreover, letting

$$Vol(u_\infty) = \int_M u_\infty^{2^*}\, d\mu_0,\ \overline{Vol}(\bar{u}) = \int_{\mathbb{R}^n} \bar{u}^{2^*}\, dx$$

for brevity, as $k \to \infty$ we have

$$1 = Vol(u_k) \to Vol(u_\infty) + L\ \overline{Vol}(\bar{u}). \tag{4.25}$$

By the result of Gidas-Ni-Nirenberg and Obata cited in Remark 3.2, we may assume that

$$\bar{u}(x) = \frac{(4n(n-1)/s_\infty)^{\frac{n-2}{4}}}{(1+|x|^2)^{\frac{n-2}{2}}}\,.$$

In particular, $\bar{u}$ achieves the Yamabe constant on $\mathbb{R}^n$, which is conformal to S^n. Letting

$$\bar{S}(v) = \frac{c_n \int_{\mathbb{R}^n} |\nabla v|^2_{g_{\mathbb{R}^n}}\, dx}{\left(\int_{\mathbb{R}^n} |v|^{2^*}\, dx\right)^{\frac{n-2}{n}}},$$

hence we find the relation

$$\bar{S}(\bar{u}) = Y(S^n, g_{S^n})\,.$$

Upon multiplying Equation (4.24) by $\bar{u}$ and integrating by parts we then obtain the identity

$$Y(S^n, g_{S^n}) = \bar{S}(\bar{u}) = s_\infty \left(\overline{Vol}(\bar{u})\right)^{1-2/2^*} = s_\infty \left(\overline{Vol}(\bar{u})\right)^{2/n}.$$

Similarly, if $u_\infty \neq 0$ we may multiply Equation (4.23) by u_∞ and integrate by parts to obtain the identity

$$S(u_\infty) = s_\infty \left(Vol(u_\infty)\right)^{2/n}.$$

Solving for $Vol(u_\infty)$ and $\overline{Vol}(\bar{u})$, from (4.25) we then obtain

$$1 = Vol(u_\infty) + L\,\overline{Vol}(\bar{u}) = \left(\frac{S(u_\infty)}{s_\infty}\right)^{n/2} + L\left(\frac{Y(S^n, g_{S^n})}{s_\infty}\right)^{n/2}, \tag{4.26}$$

that is,

$$s_\infty = \left(S(u_\infty)^{n/2} + L\, Y(S^n, g_{S^n})^{n/2}\right)^{2/n}.$$

But by definition of the Yamabe invariant we have $S(u_\infty) \geq Y(M, g_0)$; moreover, assumption (4.16) and (4.19) imply the estimate

$$s_\infty \leq s_0 \leq \left(Y(M, g_0)^{n/2} + Y(S^n, g_{S^n})^{n/2}\right)^{2/n}.$$

Thus, we either have $s_\infty = s_0$ and g_0 is a Yamabe metric, or $L = 0$; in each case the sequence (u_k) is H^1-compact with limit u_∞ inducing a Yamabe metric. Note that $u_\infty > 0$ by the maximum principle.

On the other hand, if $u_\infty \equiv 0$, from (4.16) and (4.19) we obtain

$$L^{2/n} Y(S^n, g_{S^n}) = s_\infty < s_0 \leq \left(Y(M, g_0)^{n/2} + Y(S^n, g_{S^n})^{n/2}\right)^{2/n}.$$

Since $Y(M, g_0) \leq Y(S^n, g_{S^n})$ we then conclude that $L = 1$; that is, at most a single simple bubble develops from (u_k) as $k \to \infty$. Moreover, from (4.26) in this case we have

$$1 = \left(\frac{Y(S^n, g_{S^n})}{s_\infty}\right)^{n/2}$$

and it also follows that $s_\infty = Y(S^n, g_{S^n}) = n(n-1)$.

The Compact Case $u_\infty > 0$

Throughout this part of the argument we follow Brendle [1], Section 6. The operator

$$\varphi \to A_\infty \varphi = u_\infty^{-\frac{4}{n-2}} L_0 \varphi$$

is symmetric with respect to the inner product

$$(\varphi, \psi)_{L^2(M,h_\infty)} = \int_M u_\infty^{\frac{4}{n-2}} \varphi\psi \, d\mu_0 \,.$$

The spectral theorem therefore yields the existence of a complete $L^2(M, h_\infty)$-orthonormal sequence of eigenfunctions φ_i, $i \in \mathbb{N}$, of the operator A_∞, with eigenvalues $0 < \lambda_i \le \lambda_{i+1} \to \infty$ as $i \to \infty$. Let i_0 be the first index such that $\lambda_{i_0} > s_\infty \frac{n+2}{n-2}$ and set $Z = span\{\varphi_i;\, i < i_0\}$.

Recalling the equation

$$-c_n \Delta_0 u_\infty + R_0 u_\infty - s_\infty u_\infty^{\frac{n+2}{n-2}} = u_\infty^{\frac{4}{n-2}} \big(A_\infty u_\infty - s_\infty u_\infty\big) = 0 \,,$$

by the implicit function theorem we can find a radius $\rho > 0$ such that for all $z = z^i \varphi_i \in Z$ with $|z| := ||z||_{L^2(M,h_\infty)} < \rho$ there exists a unique function $u_z = u_\infty + z + o(|z|)$ with

$$(u_z - u_\infty, \varphi_i)_{L^2(M,h_\infty)} = \int_M u_\infty^{\frac{4}{n-2}} (u_z - u_\infty)\varphi_i \, d\mu_0 = z^i,\ 1 \le i < i_0,$$

such that

$$\begin{aligned} &(A_\infty u_z - s_\infty \big(\frac{u_z}{u_\infty}\big)^{\frac{4}{n-2}} u_z, \varphi_i)_{L^2(M,h_\infty)} \\ &\qquad = \int_M \left(L_0 u_z - s_\infty u_z^{\frac{n+2}{n-2}} \right) \varphi_i \, d\mu_0 = 0 \end{aligned} \tag{4.27}$$

for all $i \ge i_0$.

Moreover, the map $z \to u_z$ is real analytic. According to results of Lojasiewicz [1], [2], there then exists a number $0 < \gamma < 1$ such that

$$S(u_z) - S(u_\infty) \le C \sup_{i<i_0} \left| \frac{\partial}{\partial z^i} S(u_z) \right|^{1+\gamma} ; \tag{4.28}$$

see also Simon [2], Equation (2.4) on p. 538. Observe that (4.27) implies the identity

$$\frac{\partial}{\partial z^i} S(u_z) = \frac{2}{Vol(u_z)^{\frac{n-2}{n}}} \int_M \left(L_0 u_z - s_\infty u_z^{\frac{n+2}{n-2}} \right) \varphi_i \, d\mu_0 - I$$

for any $i < i_0$, with error

$$I = 2 \left(\frac{S(u_z)}{Vol(u_z)} - \frac{s_\infty}{Vol(u_z)^{\frac{n-2}{n}}} \right) \int_M u_z^{\frac{n+2}{n-2}} \frac{\partial}{\partial z^i} u_z \, d\mu_0 \,.$$

Expanding

$$\left(\frac{S(u_z)}{Vol(u_z)} - \frac{s_\infty}{Vol(u_z)^{\frac{n-2}{n}}}\right) = Vol(u_z)^{\frac{2-2n}{n}} \int_M \left(L_0 u_z - s_\infty u_z^{\frac{n+2}{n-2}}\right) u_z \, d\mu_0 \,,$$

and recalling (4.27), from (4.28) we can thus estimate

$$S(u_z) - S(u_\infty) \le C \sup_{i<i_0} \left| \int_M \left(L_0 u_z - s_\infty u_z^{\frac{n+2}{n-2}}\right) \varphi_i \, d\mu_0 \right|^{1+\gamma} . \tag{4.29}$$

4.20 Normalization. For any $k \in \mathbb{N}$ we choose $z_k \in Z$ such that

$$\int_M (c_n |\nabla(u_{z_k} - u_k)|_0^2 + R_0 |u_{z_k} - u_k|^2) \, d\mu_0 = \min_{|z|<\rho} \int_M (c_n |\nabla(u_z - u_k)|_0^2 + R_0 |u_z - u_k|^2) \, d\mu_0 \,.$$

Note that H^1-convergence $u_k \to u_\infty$ implies that $z_k \to 0$ as $k \to \infty$. Decomposing

$$u_k = u_{z_k} + w_k \,,$$

now

$$\int_M L_0 w_k \frac{\partial}{\partial z^i} u_z \, d\mu_0 = \int_M (-c_n \Delta_0 w_k + R_0 w_k) \frac{\partial}{\partial z^i} u_z \, d\mu_0 = 0$$

for any $i < i_0$ by choice of z_k. Thus with error $o(1) \to 0$ as $k \to \infty$ we have

$$\begin{aligned} \lambda_i \int_M u_\infty^{\frac{4}{n-2}} \varphi_i w_k \, d\mu_0 &= \int_M L_0 \varphi_i w_k \, d\mu_0 \\ &= \int_M L_0 w_k (\varphi_i - \frac{\partial}{\partial z^i} u_z) \, d\mu_0 = o(1) ||w_k||_{H^1} \end{aligned} \tag{4.30}$$

for any such i.

4.21 Lemma. *There exist constants $c > 0$, k_0 such that for $k \ge k_0$ there holds*

$$\frac{n+2}{n-2} s_\infty \int_M u_\infty^{\frac{4}{n-2}} w_k^2 \, d\mu_0 \le (1-c) \int_M (c_n |\nabla w_k|_0^2 + R_0 |w_k|^2) \, d\mu_0 \,.$$

Proof. Otherwise for a subsequence $k \to \infty$ we may rescale $\tilde{w}_k = a_k w_k$ so that

$$1 = \int_M (c_n |\nabla \tilde{w}_k|_0^2 + R_0 |\tilde{w}_k|^2) \, d\mu_0 \le \liminf_{k\to\infty} \left(\frac{n+2}{n-2} s_\infty \int_M u_\infty^{\frac{4}{n-2}} \tilde{w}_k^2 \, d\mu_0 \right) .$$

A subsequence $\tilde{w}_k \to \tilde{w} \ne 0$ weakly in H^1, where

$$\int_M (c_n |\nabla \tilde{w}|_0^2 + R_0 |\tilde{w}|^2) \, d\mu_0 \le \frac{n+2}{n-2} s_\infty \int_M u_\infty^{\frac{4}{n-2}} \tilde{w}^2 \, d\mu_0 \,.$$

But (4.30) implies that

$$\int_M (c_n|\nabla\tilde{w}|_0^2 + R_0|\tilde{w}|^2)\,d\mu_0 \geq \lambda_{i_0}\int_M u_\infty^{\frac{4}{n-2}}\tilde{w}^2\,d\mu_0\,,$$

and a contradiction results to the choice of i_0. □

4.22 Lemma. *There exist constants $C > 0$, k_0 such that for $k \geq k_0$ there holds*

$$||w_k||_{H^1} \leq C\Big(\int_{M\times\{t_k\}} |R - s_\infty|^{\frac{2n}{n+2}}\,d\mu\Big)^{\frac{n+2}{2n}}\,.$$

Proof. By (4.30) there exist functions v_k with $(v_k,\varphi_i)_{L^2(M,h_\infty)} = 0$ for all $i < i_0$ such that $||v_k - w_k||_{H^1} = o(1)||w_k||_{H^1}$, where $o(1) \to 0$ as $k \to \infty$. Recalling the definition of u_z, we then have

$$\int_M (L_0 u_{z_k} - s_\infty u_{z_k}^{\frac{n+2}{n-2}})v_k\,d\mu_0 = 0\,.$$

Subtracting the term

$$L_0 u_k - s_\infty u_k^{\frac{n+2}{n-2}} = (R_{g(t_k)} - s_\infty)u_k^{\frac{n+2}{n-2}}$$

in the integrand, from Lemma 4.21 with a uniform constant $c > 0$ we obtain

$$\begin{aligned} c||w_k||_{H^1}^2 &\leq \int_M (L_0 w_k - \frac{n+2}{n-2}s_\infty u_\infty^{\frac{4}{n-2}}w_k)w_k\,d\mu_0 \\ &= \int_M (L_0 w_k - \frac{n+2}{n-2}s_\infty u_\infty^{\frac{4}{n-2}}w_k)v_k\,d\mu_0 + o(1)||w_k||_{H^1}^2 \\ &= \int_M \big((R_{g(t_k)} - s_\infty)u_k^{\frac{n+2}{n-2}} + d_k\big)v_k\,d\mu_0 + o(1)||w_k||_{H^1}^2 =: I_k\,, \end{aligned}$$

where

$$\begin{aligned} d_k &= s_\infty(u_k^{\frac{n+2}{n-2}} - u_{z_k}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}u_\infty^{\frac{4}{n-2}}w_k) \\ &= s_\infty\frac{n+2}{n-2}(u_{z_k}^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}})w_k + s_\infty(u_k^{\frac{n+2}{n-2}} - u_{z_k}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}u_{z_k}^{\frac{4}{n-2}}w_k)\,. \end{aligned}$$

Since we assume that $n \leq 5$ this term can be bounded

$$|d_k| \leq C(u_k + |w_k| + u_\infty)^{\frac{6-n}{n-2}}(|z_k| + |w_k|)|w_k|\,.$$

By Sobolev's embedding $H^1 \hookrightarrow L^{2^*}$, thus we can estimate

$$\int_M d_k v_k\,d\mu_0 \leq o(1)||w_k||_{H^1}^2\,.$$

It follows that

$$I_k \le C\Big(\int_{M\times\{t_k\}} |R-s_\infty|^{\frac{2n}{n+2}} u_k^{2^*}\,d\mu_0\Big)^{\frac{n+2}{2n}} ||v_k||_{L^{2^*}} + o(1)||w_k||_{H^1}^2$$
$$\le C\Big(\int_{M\times\{t_k\}} |R-s_\infty|^{\frac{2n}{n+2}}\,d\mu\Big)^{\frac{n+2}{2n}} ||w_k||_{H^1} + o(1)||w_k||_{H^1}^2\,,$$

and hence the claim. □

Proof of Proposition 4.16 in the case $u_\infty > 0$. Expand

$$L_0 u_{z_k} - s_\infty u_{z_k}^{\frac{n+2}{n-2}} = L_0 u_k - R_{g(t_k)} u_k^{\frac{n+2}{n-2}}$$
$$+ (R_{g(t_k)} - s_\infty) u_k^{\frac{n+2}{n-2}} - L_0 w_k - s_\infty (u_{z_k}^{\frac{n+2}{n-2}} - u_k^{\frac{n+2}{n-2}})\,.$$

Recalling (4.1) and observing the estimates

$$\int_M L_0 w_k \varphi_i\,d\mu_0 = \int_M w_k L_0\varphi_i\,d\mu_0$$
$$= \lambda_i \int_M u_\infty^{\frac{4}{n-2}} w_k\varphi_i\,d\mu_0 \le C||w_k||_{L^{2^*}} \le C||w_k||_{H^1}\,,\ 1\le i<i_0\,,$$

together with

$$\int_M s_\infty (u_{z_k}^{\frac{n+2}{n-2}} - u_k^{\frac{n+2}{n-2}})\varphi_i\,d\mu_0 \le C\int_M (u_k+|w_k|)^{\frac{4}{n-2}} |w_k||\varphi_i|\,d\mu_0$$
$$\le C||w_k||_{L^{2^*}} \le C||w_k||_{H^1}\,,$$

for any $i<i_0$ we have the bound

$$\Big|\int_M (L_0 u_{z_k} - s_\infty u_{z_k}^{\frac{n+2}{n-2}})\varphi_i\,d\mu_0\Big| \le \Big|\int_M (R_{g(t_k)} - s_\infty) u_k^{\frac{n+2}{n-2}}\varphi_i\,d\mu_0\Big| + C||w_k||_{H^1}\,.$$

Lemma 4.22 and (4.29) then yield the estimate

$$S(u_{z_k}) - S(u_\infty) \le C\Big(\int_{M\times\{t_k\}} |R-s_\infty|^{\frac{2n}{n+2}}\,d\mu\Big)^{\frac{n+2}{2n}(1+\gamma)}\,. \tag{4.31}$$

Likewise we expand

$$L_0 u_k u_k = L_0(u_{z_k}+w_k)(u_{z_k}+w_k)$$
$$= L_0 u_{z_k} u_{z_k} + 2L_0 u_k w_k - L_0 w_k w_k$$

and use (4.1) to conclude

$$S(u_k) = \int_M L_0(u_{z_k}+w_k)(u_{z_k}+w_k)\,d\mu_0$$
$$= \int_M L_0 u_{z_k} u_{z_k}\,d\mu_0 + 2\int_M R_{g(t_k)} u_k^{\frac{n+2}{n-2}} w_k\,d\mu_0 - \int_M L_0 w_k w_k\,d\mu_0\,.$$

We may write the latter in the form

$$S(u_k) = s_\infty + 2\int_M (R_{g(t_k)} - s_\infty) u_k^{\frac{n+2}{n-2}} w_k \, d\mu_0 \\ - \int_M (L_0 w_k w_k - \frac{n+2}{n-2} s_\infty u_{z_k}^{\frac{4}{n-2}} w_k^2) \, d\mu_0 + I_k \,,$$

where

$$I_k = (S(u_{z_k}) - s_\infty) Vol(u_{z_k})^{\frac{n-2}{n}} + s_\infty (Vol(u_{z_k})^{\frac{n-2}{n}} - 1) \\ + s_\infty \int_M (2u_k^{\frac{n+2}{n-2}} w_k - \frac{n+2}{n-2} u_{z_k}^{\frac{4}{n-2}} w_k^2) \, d\mu_0 \,.$$

By concavity of the function $s \to s^{\frac{n-2}{n}}$ we have

$$Vol(u_{z_k})^{\frac{n-2}{n}} - 1 \le \frac{n-2}{n}(Vol(u_{z_k}) - 1) = \frac{n-2}{n} \int_M (u_{z_k}^{2^*} - u_k^{2^*}) \, d\mu_0 \,.$$

Therefore we can bound the error term

$$\begin{aligned} I_k &\le (S(u_{z_k}) - s_\infty) Vol(u_{z_k})^{\frac{n-2}{n}} \\ &\quad + s_\infty \int_M (\frac{n-2}{n} u_{z_k}^{2^*} - \frac{n+2}{n-2} u_{z_k}^{\frac{4}{n-2}} w_k^2 + 2u_k^{\frac{n+2}{n-2}} w_k - \frac{n-2}{n} u_k^{2^*}) \, d\mu_0 \\ &= (S(u_{z_k}) - S(u_\infty)) Vol(u_{z_k})^{\frac{n-2}{n}} \\ &\quad - \frac{2s_\infty}{2^*} \int_M (u_k^{2^*} - 2^* u_k^{\frac{n+2}{n-2}} w_k + 2^* \frac{n+2}{n-2} u_{z_k}^{\frac{4}{n-2}} w_k^2/2 - u_{z_k}^{2^*}) \, d\mu_0 \,. \end{aligned}$$

Recalling that $u_k = u_{z_k} + w_k$, we can bound

$$|u_k^{2^*} - 2^* u_k^{\frac{n+2}{n-2}} w_k + 2^* \frac{n+2}{n-2} u_{z_k}^{\frac{4}{n-2}} w_k^2/2 - u_{z_k}^{2^*}| \le C(u_k + |w_k|)^{\frac{6-n}{n-2}} |w_k|^3 \,.$$

In view of (4.31) and Lemma 4.22 we thus obtain the error estimate

$$\begin{aligned} I_k &\le C\Big(\int_{M\times\{t_k\}} |R - s_\infty|^{\frac{2n}{n+2}} \, d\mu\Big)^{\frac{n+2}{2n}(1+\gamma)} + C||w_k||_{H^1}^3 \\ &\le C\Big(\int_{M\times\{t_k\}} |R - s_\infty|^{\frac{2n}{n+2}} \, d\mu\Big)^{\frac{n+2}{2n}(1+\gamma)} \,. \end{aligned}$$

By Hölder's inequality and Lemma 4.22 we also may bound

$$\begin{aligned} \int_M (R_{g(t_k)} - s_\infty) u_k^{\frac{n+2}{n-2}} w_k \, d\mu_0 &\le \Big(\int_{M\times\{t_k\}} |R - s_\infty|^{\frac{2n}{n+2}} \, d\mu\Big)^{\frac{n+2}{2n}} ||w_k||_{L^{2^*}} \\ &\le C\Big(\int_{M\times\{t_k\}} |R - s_\infty|^{\frac{2n}{n+2}} \, d\mu\Big)^{\frac{n+2}{n}} \,, \end{aligned}$$

while by Lemma 4.21 with a uniform constant $c > 0$ for $k \ge k_0$ we have

$$\int_M (L_0 w_k w_k - \frac{n+2}{n-2} s_\infty u_{z_k}^{\frac{4}{n-2}} w_k^2) \, d\mu_0 \ge c||w_k||_{H^1}^2 \,.$$

Finally, we can estimate

$$\left(\int_{M\times\{t_k\}} |R-s_\infty|^{\frac{2n}{n+2}}\,d\mu\right)^{\frac{n+2}{2n}} \le F_{\frac{2n}{n+2}}(g(t_k))^{\frac{n+2}{2n}} + s(t_k) - s_\infty\,.$$

Combining the previous estimates and observing that $\gamma < 1$, we obtain

$$s(t_k) - s_\infty = S(u_k) - s_\infty \le CF_{\frac{2n}{n+2}}(g(t_k))^{\frac{n+2}{2n}(1+\gamma)} + C(s(t_k)-s_\infty)^{1+\gamma}\,.$$

Since $s(t_k) \to s_\infty$ as $k \to \infty$, our claim in Proposition 4.16 follows. □

The idea of using the Lojasiewicz inequality for proving convergence of a geometric flow was pioneered by Leon Simon [1].

Bubbling: The Case $u_\infty \equiv 0$.

In order to deal with "bubbling" the key idea is to make an optimally balanced choice for the position x_k, scale r_k, and size α_k of the "bubble" separating from the sequence (u_k) by Lemma 4.19, and to use the positive mass theorem.

4.23 The standard bubble. Following Lee-Parker [1], Theorem 5.1, or Günther [2] we introduce local conformal normal coordinates on (M, g_0). Fix some number $N \in \mathbb{N}$. Given a point $x \in M$, we can find a conformal metric $h_x = \varphi_x^{\frac{4}{n-2}} g_0$ with $\varphi_x(x) = 1$ such that

$$det(D\,exp_x(\xi)) = 1 + O(|\xi|^N)$$

near $\xi = 0$, where $D\,exp_x(\xi)$ denotes the differential of the exponential map in the metric h_x at a point $\xi \in T_xM$. Moreover, we have

$$|R_{h_x}(y)| \le C\rho_x(y)^2\,,$$

where $\rho_x(y)$ denotes the Riemannian distance from x to y as measured by h_x. Also let G_x be the Green's function of the operator L_{h_x} with pole at x, satisfying

$$L_{h_x}G_x = -c_n\Delta_{h_x}G_x + R_{h_x}G_x = 0 \text{ for } y \ne x\,.$$

Moreover, G_x satisfies the estimates

$$|G_x(y) - \rho_x(y)^{2-n} - A_x| \le C\rho_x(y)\,,\ |d(G_x(y) - \rho_x(y)^{2-n})| \le C\,,$$

with a constant $A_x > 0$ by the positive mass theorem. As shown by Habermann-Jost [1], Proposition I.1.3, the term A_x smoothly depends on $x \in M$, and we have $\inf_M A_x = A > 0$.

Fix a smooth cut-off function χ such that $0 \le \chi \le 1$, $\chi(s) = 1$ for $s \le 1$, $\chi(s) = 0$ for $s \ge 2$. For any small number $\delta > 0$ scale $\chi_\delta(s) = \chi(s/\delta)$. For $x \in M$, $\varepsilon > 0$ then let $u^*_{x,\varepsilon}(y) = \varphi_x(y)U^*_{x,\varepsilon}(y)$, where similar to (2.5) the profile $U^*_{x,\varepsilon}$ is given by

$$U^*_{x,\varepsilon}(y) = \left(\frac{4n(n-1)\varepsilon^2}{s_\infty}\right)^{\frac{n-2}{4}} \left(\frac{\chi_\delta(\rho_x(y))}{(\varepsilon^2+\rho_x(y)^2)^{\frac{n-2}{2}}} + \big(1-\chi_\delta(\rho_x(y))\big)G_x(y)\right).$$

Then we have the following result of Schoen [1].

4.24 Theorem. *Suppose that $3 \le n \le 5$ and that (M, g_0) is not conformally equivalent to the standard sphere S^n. Then, with constants C, $c > 0$ independent of $x \in M$, $\delta > 0$, and $\varepsilon > 0$ there holds*

$$S(u^*_{x,\varepsilon}) \le Y(S^n, g_{S^n}) - c\varepsilon^{n-2} + C\delta\varepsilon^{n-2} + C\delta^{-n}\varepsilon^n \,.$$

Proof. A detailed proof, adapting the argument of Schoen [1], thereby using the uniform positivity of the mass, is given by Brendle [1], Proposition B.3. □

By Theorem 4.24 we can fix $\delta > 0$ such that for sufficiently small $\varepsilon > 0$ we have $S(u^*_{x,\varepsilon}) < Y(S^n, g_{S^n})$. To be consistent with the notation of Lemma 4.19 henceforth we denote the scale as $r > 0$ and we let $u^*_{x,r}$ denote a standard bubble of scale $r > 0$ centered at x. Then Lemma 4.19 provides a decomposition

$$u_k = \alpha_k u^*_{x_k, r_k} + w_k =: v_k + w_k$$

with suitable x_k, $r_k > 0$ and $\alpha_k \to 1$. In fact, similar to the approach in the case when $u_\infty > 0$ we now normalize this choice by the requirement that

$$\int_M (c_n|\nabla(u_k - \alpha_k u^*_{x_k,r_k})|_0^2 + R_0|u_k - \alpha_k u^*_{x_k,r_k}|^2)\, d\mu_0$$
$$= \min_{\alpha>0,\, x\in M,\, r>0} \int_M (c_n|\nabla(u_k - \alpha u^*_{x,r})|_0^2 + R_0|u_k - \alpha u^*_{x,r}|^2)\, d\mu_0 \,.$$

Then we have the following result similar to (4.30).

4.25 Lemma. *With error $o(1) \to 0$ as $k \to \infty$ there holds*

$$\int_M v_k^{\frac{n+2}{n-2}} \psi_k w_k \, d\mu_0 = o(1)||w_k||_{H^1}$$

for the functions

$$\psi_k = 1 \,,\ \psi_k = \frac{r_k^2}{r_k^2 + d(x, x_k)^2} \,,\ \psi_k = \frac{r_k exp_{x_k}^{-1}(x)}{r_k^2 + d(x, x_k)^2}$$

related to the generators of the family $(\alpha u^*_{x,r})_{\alpha>0,\, x\in M,\, r>0}$.

Proof. Observe that

$$L_0 v_k = L_0 u_k - L_0 w_k = R_{g(t_k)} u_k^{\frac{n+2}{n-2}} - L_0 w_k = s_\infty v_k^{\frac{n+2}{n-2}} + d_k$$

with

$$d_k = (R_{g(t_k)} - s_\infty) u_k^{\frac{n+2}{n-2}} + s_\infty (u_k^{\frac{n+2}{n-2}} - v_k^{\frac{n+2}{n-2}}) - L_0 w_k \to 0 \text{ in } H^{-1} \,.$$

By minimality of the above decomposition with respect to α then we have

$$0 = \int_M L_0 v_k w_k \, d\mu_0 = s_\infty \int_M v_k^{\frac{n+2}{n-2}} w_k \, d\mu_0 + o(1)||w_k||_{H^1} \,.$$

as claimed. The other assertions follow similarly. □

The following result provides the analogue of Lemma 4.21. It reflects the fact that the functions u_ε^* defined in (2.5) up to scaling and translation strictly minimize the Sobolev ratio $S(u)$ on $\mathbb{R}^n$.

4.26 Lemma. *There exist constants $c > 0$, k_0 such that for $k \geq k_0$ there holds*

$$\frac{n+2}{n-2} s_\infty \int_M v_k^{\frac{4}{n-2}} w_k^2 \, d\mu_0 \leq (1-c) \int_M (c_n |\nabla w_k|_0^2 + R_0 |w_k|^2) \, d\mu_0 \,.$$

Proof. Otherwise we rescale $\tilde{w}_k = a_k w_k$ so that

$$1 = \int_M (c_n |\nabla \tilde{w}_k|_0^2 + R_0 |\tilde{w}_k|^2) \, d\mu_0 \leq \frac{n+2}{n-2} s_\infty \int_M v_k^{\frac{4}{n-2}} \tilde{w}_k^2 \, d\mu_0 + o(1) \tag{4.32}$$

with error $o(1) \to 0$ for a subsequence $k \to \infty$. Letting

$$\hat{w}_k(\xi) = r_k^{\frac{n-2}{2}} \tilde{w}_k(exp_{x_k}(r_k \xi)) \colon B_{R/r_k}(0) \subset T_{x_k} M \to \mathbb{R} \,,$$

for some $R < \iota_0$, the injectivity radius of (M, g_0), by Sobolev's inequality we have

$$Y_0 \bigg(\int_{B_{R/r_k}(0)} |\hat{w}_k|^{2^*} \, d\xi \bigg)^{\frac{n-2}{n}} \leq 1 + o(1) \,, \ \int_{B_{R/r_k}(0)} c_n |\nabla \hat{w}_k|^2 \, d\xi \leq 1 + o(1) \,,$$

and a subsequence $\hat{w}_k \to \hat{w}$ weakly in $H^1_{loc}(\mathbb{R}^n)$, where $\hat{w} \in D^{1,2}(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \frac{\hat{w}^2}{(1+|\xi|^2)^2} \, d\xi > 0 \,.$$

Moreover, passing to the limit in (4.32) and simplifying the factors, we find

$$\int_{\mathbb{R}^n} |\nabla \hat{w}|^2 \, d\xi \leq n(n+2) \int_{\mathbb{R}^n} \frac{\hat{w}^2}{(1+|\xi|^2)^2} \, d\xi \,.$$

But by Lemma 4.25 there holds

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^{\frac{n+2}{2}}} \, \hat{w} \, d\xi &= 0 \,, \\
\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^{\frac{n+2}{2}}} \, \frac{1-|\xi|^2}{1+|\xi|^2} \, \hat{w} \, d\xi &= 0 \,, \\
\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^{\frac{n+2}{2}}} \, \frac{\xi}{1+|\xi|^2} \, \hat{w} \, d\xi &= 0 \,.
\end{aligned}$$

A result of Rey [1], Appendix D, pp. 49–51, now implies that $\hat{w} \equiv 0$, which gives the desired contradiction. □

The proof of Proposition 4.16 now can be completed just as in the case when $u_\infty > 0$.

Proof of Proposition 4.16 in the case $u_\infty \equiv 0$. Expand

$$\begin{aligned} S(u_k) &= \int_M L_0(v_k + w_k)(v_k + w_k)\, d\mu_0 \\ &= \int_M L_0 v_k v_k\, d\mu_0 + 2\int_M R_{g(t_k)} u_k^{\frac{n+2}{n-2}} w_k\, d\mu_0 - \int_M L_0 w_k w_k\, d\mu_0 \\ &= s_\infty + 2\int_M (R_{g(t_k)} - s_\infty) u_k^{\frac{n+2}{n-2}} w_k\, d\mu_0 \\ &\quad - \int_M (L_0 w_k w_k - \frac{n+2}{n-2} s_\infty v_k^{\frac{4}{n-2}} w_k^2)\, d\mu_0 + I_k\,, \end{aligned}$$

where the error term

$$\begin{aligned} I_k = (S(v_k) - s_\infty) Vol(v_k)^{\frac{n-2}{n}} + s_\infty (Vol(v_k)^{\frac{n-2}{n}} - 1) \\ + s_\infty \int_M (2u_k^{\frac{n+2}{n-2}} w_k - \frac{n+2}{n-2} v_k^{\frac{4}{n-2}} w_k^2)\, d\mu_0 \end{aligned}$$

in view of the estimate $S(v_k) = S(u^*_{x_k, r_k}) \leq Y(S^n, g_{S^n}) = s_\infty$ may be bounded

$$I_k \leq -\frac{2s_\infty}{2^*} \int_M (u_k^{2^*} - 2^* u_k^{\frac{n+2}{n-2}} w_k + 2^* \frac{n+2}{n-2} v_k^{\frac{4}{n-2}} w_k^2/2 - v_k^{2^*})\, d\mu_0 \leq C||w_k||^3_{H^1}$$

similar to the case when $u_\infty > 0$. Since Lemmas 4.26 and 4.19 yield that

$$||w_k||^2_{H^1} \leq C \int_M (L_0 w_k w_k - \frac{n+2}{n-2} s_\infty v_k^{\frac{4}{n-2}} w_k^2)\, d\mu_0$$

for $k \geq k_0$, the claim follows from Hölder's inequality as before. □

4.27 Remarks. (1°) Theorems 4.9 and 4.11 show that it may not always be the best strategy to follow the gradient flow for a variational problem; sometimes an apparently singular (possibly degenerate) evolution equation may be much more tractable, due in this case to the intrinsic geometric meaning of (4.4). We encounter a similar phenomenon in Section 6.
(2°) An argument analogous to Ye's proof above works in $n = 2$ dimensions; see Bartz-Struwe-Ye [1]. See also Struwe [26] for a new approach in this case.
(3°) For further material and references on the Yamabe problem we refer the reader to the survey by Lee-Parker [1].
(4°) See Kazdan-Warner [1], Aubin [3], Bahri-Coron [1], Schoen [2], Chang-Yang [1], and - more recently - Struwe [27] for the related problem of finding conformal metrics of prescribed scalar curvature ("Nirenberg's problem").
(5°) Another variant of the Yamabe problem is the singular Yamabe problem of finding complete conformal metrics of constant scalar curvature, for instance, on $S^n \setminus \Gamma$, where Γ is a smooth k-dimensional submanifold of S^n. If $k > \frac{n-2}{2}$, this leads to a coercive problem, reminiscent of the case $S(M) < 0$ for the Yamabe problem. This was solved by Loewner-Nirenberg [1]. The case $k < \frac{n-2}{2}$ is more difficult and was solved only recently by Mazzeo-Smale [1], Mazzeo-Pacard [1].

5. The Dirichlet Problem for the Equation of Constant Mean Curvature

Another borderline case of a variational problem is the following: Let Ω be a bounded domain in $\mathbb{R}^2$ with generic point $z = (x, y)$ and let $u_0 \in C^0(\overline{\Omega}; \mathbb{R}^3)$, $H \in \mathbb{R}$ be given. Find a solution $u \in C^2(\Omega; \mathbb{R}^3) \cap C^0(\overline{\Omega}; \mathbb{R}^3)$ to the problem

$$\Delta u = 2H u_x \wedge u_y \qquad \text{in } \Omega \ , \tag{5.1}$$

$$u = u_0 \qquad \text{on } \partial\Omega \ . \tag{5.2}$$

Here, for $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{R}^3$, $a \wedge b$ denotes the wedge product $a \wedge b = (a_2 b_3 - b_2 a_3, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2)$ and, for instance, $u_x = \frac{\partial}{\partial x} u$. (5.1) is the equation satisfied by surfaces of mean curvature H in conformal representation.

Surprisingly, (5.1) is of variational type. In fact, solutions of (5.1) may arise as "soap bubbles", that is, surfaces of least area enclosing a given volume. Also for prescribed Dirichlet data, where a geometric interpretation of (5.1) is impossible, we may recognize (5.1) as the Euler-Lagrange equations associated with the variational integral

$$E_H(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dz + \frac{2H}{3} \int_\Omega u \cdot u_x \wedge u_y \, dz \ .$$

For smooth "surfaces" u, the term

$$V(u) := \frac{1}{3} \int_\Omega u \cdot u_x \wedge u_y \, dz$$

may be interpreted as the algebraic volume enclosed between the "surface" parametrized by u and a fixed reference surface spanning the "curve" defined by the Dirichlet data u_0; see Figure 5.1.

Indeed, computing the variation of the volume V at a point $u \in C^2(\Omega; \mathbb{R}^3)$ in direction of a vector $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$, we obtain

$$\begin{aligned} 3 \frac{d}{d\varepsilon} V(u + \varepsilon\varphi)|_{\varepsilon=0} &= \int_\Omega \big(\varphi \cdot u_x \wedge u_y + u \cdot \varphi_x \wedge u_y + u \cdot u_x \wedge \varphi_y \big) \, dz \\ &= 3 \int_\Omega \varphi \cdot u_x \wedge u_y \, dz + \int_\Omega \varphi \cdot (u \wedge u_{yx} + u_{xy} \wedge u) \, dz \ , \end{aligned}$$

and the second integral vanishes by anti-symmetry of the wedge product. Hence critical points $u \in C^2(\Omega; \mathbb{R}^3)$ of E solve (5.1).

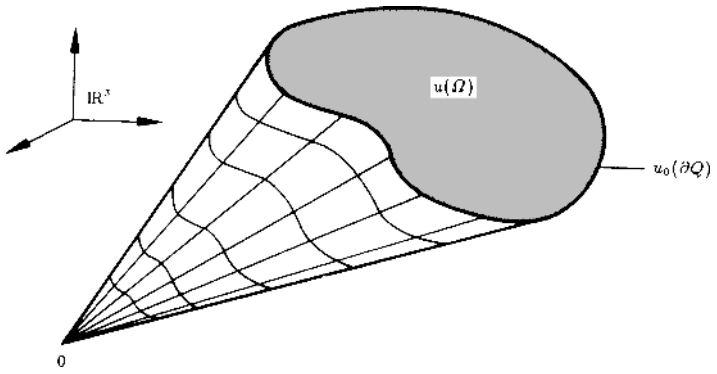

Fig. 5.1. On the volume functional

Small Solutions

Since V is cubic, in E_H the Dirichlet integral dominates if u is "small" and we can expect that for "small data" and "small" H a solution of (5.1), (5.2) can be obtained by minimizing E_H in a suitable convex set. Generalizing earlier results by Heinz [1] and Werner [1], Hildebrandt [2] has obtained the following result, which is conjectured to give the best possible bounds for the type of constraint considered:

5.1 Theorem. *Suppose* $u_0 \in H^{1,2} \cap L^\infty(\Omega; \mathbb{R}^3)$. *Then for any* $H \in \mathbb{R}$ *such that*

$$\|u_0\|_{L^\infty} \cdot |H| < 1 \ ,$$

there exists a solution $u \in u_0 + H_0^{1,2}(\Omega; \mathbb{R}^3)$ *of (5.1), (5.2) such that*

$$\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty} \ .$$

The solution u *is characterized by the condition*

$$E_H(u) = \min\{E_H(v) \ ; \ v \in u_0 + H_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^3), \ \|v\|_{L^\infty}|H| \leq 1\} \ .$$

In particular, u *is a relative minimizer of* E_H *in* $u_0 + H_0^{1,2} \cap L^\infty(\Omega; \mathbb{R}^3)$.

5.2 Remark. Working with a different geometric constraint Wente [1; Theorem 6.1] and Steffen [1; Theorem 2.2] prove the existence of a relative minimizer provided

$$E_0(u_0)H^2 < \frac{2}{3}\pi \ ,$$

where

$$E_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dz$$

is the Dirichlet integral of u. The bound is not optimal, see Struwe [18; Remark IV.4.14]; it is conjectured that $E_0(u_0)H^2 < \pi$ suffices.

Proof of Theorem 5.1. Let

$$M = \{v \in u_0 + H_0^{1,2}(\Omega;\mathbb{R}^3)\ ;\ v \in L^\infty(\Omega;\mathbb{R}^3),\ \|v\|_{L^\infty}|H| \le 1\}\ .$$

M is closed and convex, hence weakly closed in $H^{1,2}(\Omega)$. Note that for $u \in M$ we can estimate

$$|H\,u\cdot u_x \wedge u_y| \le \frac{1}{2}|H|\ \|u\|_{L^\infty}|\nabla u|^2 \le \frac{1}{2}|\nabla u|^2\ , \tag{5.3}$$

almost everywhere in Ω. Hence

$$E_H(u) \ge \frac{1}{3}E_0(u)\ ;$$

that is, E_H is coercive on M with respect to the $H^{1,2}$-norm. Moreover, by (5.3) on M the functional E_H may be represented by an integral

$$E_H(u) = \int_\Omega F(u,\nabla u)\,dx$$

where

$$F(u,p) = \frac{1}{2}|p|^2 + \frac{2}{3}Hu\cdot p_1 \wedge p_2$$

is non-negative, continuous in $u \in \mathbb{R}^3$, and convex in $p = (p_1,p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ for all $u \in \mathbb{R}^3$ with $|H||u| \le 1$. Hence from Theorem 1.6 we infer that E_H is weakly lower semi-continuous on M.

By Theorem I.1.2, therefore, E_H attains its infimum on M at a point $u \in M$. Moreover, E_H is analytic in $H^{1,2} \cap L^\infty(\Omega;\mathbb{R}^3)$. Therefore we may compute the directional derivative of E_H in the direction of any vector pointing from u into M. Let $\varphi \in C_0^\infty(\Omega)$, $0 \le \varphi \le 1$ and choose $v = u(1-\varphi) \in M$ as comparison function. Then by (5.3) we have

$$\begin{aligned}
0 &\ge \langle u - v, DE_H(u)\rangle = \langle u\varphi, DE_H(u)\rangle \\
&= \int_\Omega \big(\nabla u\nabla(u\varphi) + 2H\,u\cdot u_x\wedge u_y\varphi\big)\,dz \\
&= \int_\Omega \big(|\nabla u|^2 + 2Hu\cdot u_x\wedge u_y\big)\varphi\,dz + \frac{1}{2}\int_\Omega \nabla\big(|u|^2\big)\nabla\varphi\,dz \\
&\ge \frac{1}{2}\int_\Omega \nabla\big(|u|^2\big)\nabla\varphi\,dz\ .
\end{aligned}$$

Hence $|u|^2$ is weakly sub-harmonic on Ω. By the weak maximum principle, Theorem B.6 of Appendix B, $|u|^2$ attains its supremum on $\partial\Omega$. That is, there holds

$$\|u\|_{L^\infty} \le \|u_0\|_{L^\infty}\ ,$$

as desired. But then u lies interior to M relative to $H^{1,2} \cap L^\infty$ and $DE_H(u) = 0$; which means that u weakly solves (5.1), (5.2). By a result of Wente [1; Theorem 5.5], finally, any weak solution of (5.1) is also regular, in fact, analytic in Ω. □

In view of the cubic character of V, having established the existence of a relatively minimal solution to (5.1), (5.2), we are now led to expect the existence of a second solution for $H \neq 0$. This is also supported by geometrical evidence; see Figure 5.2.

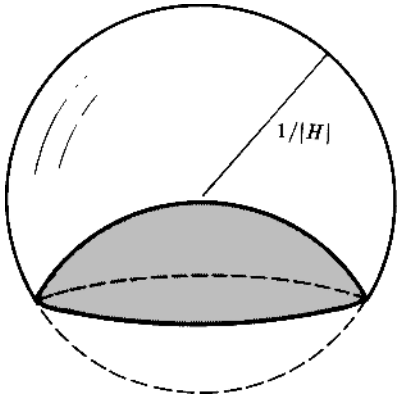

Fig. 5.2. A small and a large spherical cap of radius $1/|H|$ for $0 < |H| < 1$ give rise to distinct solutions of (5.1) with boundary data $u_0(z) = z$ on $\partial B_1(0; \mathbb{R}^2)$.

We start with an analysis of V.

The Volume Functional

In the preceding theorem we have used the obvious fact that V is smooth on $H^{1,2} \cap L^\infty(\Omega; \mathbb{R}^3)$ – but much more is true. Without proof we state the following result due to Wente [1; Section III]:

5.3 Lemma. *For any $u_0 \in H^{1,2} \cap L^\infty(\Omega; \mathbb{R}^3)$ the volume functional V extends to an analytic functional on the affine space $u_0 + H_0^{1,2}(\Omega; \mathbb{R}^3)$ and the following expansion holds:*

$$V(u_0 + \varphi) = V(u_0) + \langle \varphi, DV(u_0) \rangle + \frac{1}{2} D^2V(u_0)(\varphi, \varphi) + V(\varphi) \ . \tag{5.4}$$

Moreover, the derivatives

$$DV \colon H^{1,2}(\Omega; \mathbb{R}^3) \to H^{-1}(\Omega; \mathbb{R}^3) = H_0^{1,2}(\Omega; \mathbb{R}^3)^* \ ,$$

$$D^2V \colon H^{1,2}(\Omega; \mathbb{R}^3) \to \left(\left[H_0^{1,2}(\Omega; \mathbb{R}^3) \right]^2 \right)^*$$

are continuous and bounded in terms of the Dirichlet integral

$$\langle \varphi, DV(u)\rangle = \int_\Omega \varphi \cdot u_x \wedge u_y \; dz \le cE_0(u)E_0(\varphi)^{1/2} \;,$$
$$D^2V(u)(\varphi,\psi) = \int_\Omega u \cdot (\varphi_x \wedge \psi_y + \psi_x \wedge \varphi_y) \; dz \le c\big(E_0(u)E_0(\varphi)E_0(\psi)\big)^{1/2} \;.$$

Furthermore, DV and D^2V are weakly continuous in the sense that, if $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega;\mathbb{R}^3)$, then

$$\langle \varphi, DV(u_m)\rangle \to \langle \varphi, DV(u)\rangle, \quad \textit{for all } \; \varphi \in H_0^{1,2}(\Omega;\mathbb{R}^3) \;,$$
$$D^2V(u_m)(\varphi,\psi) \to D^2V(u)(\varphi,\psi), \quad \textit{for all } \; \varphi,\psi \in H_0^{1,2}(\Omega;\mathbb{R}^3) \;.$$

Finally, for any $u \in H^{1,2}(\Omega;\mathbb{R}^3)$ the bilinear form $D^2V(u)$ is compact; that is, if $\varphi_m \rightharpoonup \varphi$, $\psi_m \rightharpoonup \psi$ weakly in $H_0^{1,2}(\Omega;\mathbb{R}^3)$, then

$$D^2V(u)(\varphi_m,\psi_m) \to D^2V(u)(\varphi,\psi) \;.$$

Wente's proof is based on the isoperimetric inequality, Theorem 5.4 below. However, with more modern tools, we can also give an entirely analytic proof of Lemma 5.3. In fact, the special properties of V are due to the anti-symmetry of the volume form which gives rise to certain cancellation properties as in our discussion of the compensated compactness scheme; compare Section I.3. In particular, the above bounds for V and its derivatives are related to the observation of Coifman-Lions-Meyer-Semmes [1] that the cross product $\varphi_x \wedge \psi_y$ for $\varphi \;, \psi \in H_0^{1,2}(\Omega\;;\mathbb{R}^3)$ belongs to the Hardy space $\mathcal{H}^1$, with $\mathcal{H}^1$-norm bounded in terms of the Dirichlet integrals of φ and ψ. Moreover, we have $H^{1,2}(\Omega;\mathbb{R}^3) \hookrightarrow BMO(\Omega;\mathbb{R}^3)$, and, by a result of Fefferman-Stein [1], BMO is the dual space of $\mathcal{H}^1$. Hence, for instance, we derive the estimate

$$\int_\Omega u \cdot \varphi_x \wedge \psi_y \; dz \le C\|u\|_{BMO}\|\varphi_x \wedge \psi_y\|_{\mathcal{H}^1} \le C\big(E_0(u)E_0(\varphi)E_0(\psi)\big)^{1/2} \;.$$

The remaining properties of V can be derived similarly, using these techniques.

The remaining term in our functional E_H is simply Dirichlet's integral $E_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \; dz$, well familiar from I.2.7–I.2.10. Both E_0 and V are conformally invariant, in particular invariant under scaling $u \to u(Rx)$.

The fundamental estimate for dealing with the functional E is the isoperimetric inequality for closed surfaces in $\mathbb{R}^3$; see for instance Radó [2]. This inequality for (5.1), (5.2) plays the same role as the Sobolev inequality $S\|u\|_{L^{2^*}}^2 \le \|u\|_{H^{1,2}}^2$ played for problem (1.1), (1.3).

5.4 Theorem. *For any "closed surface" $\varphi \in H_0^{1,2}(\Omega;\mathbb{R}^3)$ there holds*

$$36\pi\big|V(\varphi)\big|^2 \le E_0(\varphi)^3 \;.$$

The constant 36π is best possible.

Remark. The best constant 36π is attained for instance on the function $\varphi \in D^{1,2}(\mathbb{R}^2)$,

$$\varphi(x,y) = \frac{2}{1+x^2+y^2}(x,y,1) \ ,$$

corresponding to stereographic projection of a sphere of radius 1 above $(0,0) \in \mathbb{R}^2$ onto $\mathbb{R}^2$, and its rescalings

$$\varphi_\varepsilon(x,y) = \varphi(x/\varepsilon, y/\varepsilon) = \frac{2\varepsilon}{\varepsilon^2+x^2+y^2}(x,y,\varepsilon) \ . \tag{5.5}$$

φ and φ_ε solve Equation (5.1) on $\mathbb{R}^2$ with $H = 1$: the mean curvature of the unit sphere in $\mathbb{R}^3$.

Wente's Uniqueness Result

Using the unique continuation property for the analytic Equation (5.1), analogous to Theorem 1.2 we can show that the best constant in the isoperimetric inequality is never achieved on a domain $\Omega \subseteq \mathbb{R}^2$, $\Omega \neq \mathbb{R}^2$. Moreover, similar to Theorem 1.3, a sharper result holds, due to Wente [4]:

5.5 Theorem. *If $\Omega \subset \mathbb{R}^2$ is smoothly bounded and simply connected then any solution $u \in H_0^{1,2}(\Omega;\mathbb{R}^3)$ to (5.1) vanishes identically.*

Proof. By conformal invariance of (5.1) we may assume that Ω is a ball $B_1(0;\mathbb{R}^2)$. Reflecting $u(z) = -u\left(\frac{z}{|z|^2}\right)$ we extend u as a (weak) solution of (5.1) on $\mathbb{R}^2$. From Wente's regularity result (Wente [1; Theorem 5.5]) we infer that u is smooth and solves (5.1) classically. Now by direct computation we see that the function

$$\Phi(x+iy) = \left(|u_x|^2 - |u_y|^2\right) - 2iu_x \cdot u_y$$

is holomorphic on $\mathbb{C}$. Since

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dz = 2\int_{B_1(0)} |\nabla u|^2 \, dz < \infty \ ,$$

it follows that $\Phi \in L^1(\mathbb{R}^2)$, and hence that $\Phi \equiv 0$ by the mean-value property of holomorphic functions. That is, u is conformal. But then, since $u \equiv 0$ on $\partial B_1(0;\mathbb{R}^2)$, it follows that also $\nabla u \equiv 0$ on $\partial B_1(0;\mathbb{R}^2)$ and hence, by unique continuation, that $u \equiv 0$; see Hartmann-Wintner [1; Corollary 1]. □

Theorem 5.5 – like Theorem 1.3 in the context of problem (1.1), (1.3) – proves that E_H cannot satisfy (P.-S.) globally on $H_0^{1,2}(\Omega)$, for any $H \neq 0$. Indeed, note that $E_H(0) = 0$. Moreover, by Theorem 5.4 for $u \in H_0^{1,2}(\Omega;\mathbb{R}^3)$ with $E_0(u) = \frac{4\pi}{H^2}$ there holds

$$E_H(u) = E_0(u) + 2HV(u) \geq E_0(u)\left(1 - \sqrt{\frac{4H^2}{36\pi}E_0(u)}\right)$$
$$\geq \alpha := \frac{4\pi}{3H^2} > 0 \ ,$$

while for any comparison surface u with $V(u) \neq 0$, if $HV(u) < 0$, we have

$$E_H(\rho u) = \rho^2 E_0(u) + 2H\rho^3 V(u) \to -\infty \qquad \text{as } \rho \to \infty \ .$$

Hence, if E_H satisfied (P.-S.) globally on $H_0^{1,2}(\Omega;\mathbb{R}^3)$, from Theorem II.6.1 we would obtain a contradiction to Wente's uniqueness result Theorem 5.5.

Local Compactness

However, the following analogue of Lemma 2.3 holds:

5.6 Lemma. *Suppose $u_0 \in H^{1,2} \cap L^\infty(\Omega;\mathbb{R}^3)$ is a relative minimizer of E_H in the space $u_0 + H^{1,2} \cap L^\infty(\Omega;\mathbb{R}^3)$. Then for any $\beta < E_H(u_0) + \frac{4\pi}{3H^2}$ condition (P.-S.)$_\beta$ holds on the affine space $\{u_0\} + H_0^{1,2}(\Omega;\mathbb{R}^3)$.*

For the proof of Lemma 5.6 we need $D^2E_H(u_0)$ to be positive definite on $H_0^{1,2}(\Omega;\mathbb{R}^3)$. By (5.3) this is clear for the relative minimizers constructed in Theorem 5.1, if $\|u_0\|_{L^\infty}|H| < \frac{1}{2}$. In the general case some care is needed. Also note the subtle difference in the topology of $H^{1,2} \cap L^\infty$ considered in Theorem 5.1 and $H^{1,2}$ considered here.

Postponing the proof of Lemma 5.6 for a moment we establish the following result by Brezis and Coron [2; Lemma 3]:

5.7 Lemma. *Suppose $u_0 \in H^{1,2} \cap L^\infty(\Omega;\mathbb{R}^3)$ is a relative minimizer of E_H in $u_0 + H_0^{1,2} \cap L^\infty(\Omega;\mathbb{R}^3)$. Then u_0 is a relative minimizer of E_H in $u_0 + H_0^{1,2}(\Omega;\mathbb{R}^3)$, and there exists a constant $\delta > 0$ such that*

$$D^2E_H(u_0)(\varphi,\varphi) \geq \delta\, E_0(\varphi), \quad \textit{for all } \varphi \in H_0^{1,2}(\Omega;\mathbb{R}^3) \ .$$

Proof. We may assume that $H \neq 0$. By density of $C_0^\infty(\Omega;\mathbb{R}^3)$ in $H_0^{1,2}(\Omega;\mathbb{R}^3)$, since u_0 is a relative minimizer in $u_0 + H_0^{1,2} \cap L^\infty(\Omega;\mathbb{R}^3)$ clearly we have

$$\delta = \inf\big\{D^2E_H(u_0)(\varphi,\varphi) \ ; \ \varphi \in H_0^{1,2}(\Omega;\mathbb{R}^3),\ E_0(\varphi) = 1\big\} \geq 0 \ .$$

Note that

$$D^2E_H(u_0)(\varphi,\varphi) = 2E_0(\varphi) + 2HD^2V(u_0)(\varphi,\varphi) \ ,$$

and $D^2V(u_0)$ is compact by Lemma 5.3. Hence, if

$$\nu := 2H \inf_{\varphi \in H_0^{1,2}(\Omega;\mathbb{R}^3);\ E_0(\varphi)=1} D^2V(u_0)(\varphi,\varphi) < 0 \ ,$$

then ν is attained. Indeed, a minimizing sequence (φ_m) for ν accumulates weakly in $H_0^{1,2}(\Omega;\mathbb{R}^3)$ at a limit φ with

$$2HD^2V(u_0)(\varphi,\varphi) = \nu < 0\ , E_0(\varphi) \le 1\ .$$

In particular, $\varphi \neq 0$, and we can consider the normalized function $\bar\varphi = \frac{\varphi}{\sqrt{E_0(\varphi)}}$ as comparison function with

$$2HD^2V(u_0)(\bar\varphi,\bar\varphi) = \frac{\nu}{E_0(\varphi)}\ .$$

It follows that $E_0(\varphi) = 1$; that is, ν is attained. In terms of δ, we thus conclude that, if $\delta = 2 + \nu < 2$, then δ is attained. In particular, if $\delta = 0$ there exists φ such that $E_0(\varphi) = 1$ and

$$D^2E_H(u_0)(\varphi,\varphi) = 0 = \inf\big\{D^2E_H(u_0)(\psi,\psi)\ ;\ \psi \in H_0^{1,2}(\Omega;\mathbb{R}^3),\ E_0(\psi) = 1\big\}\ .$$

Necessarily, φ satisfies the Euler equation for $D^2E_H(u_0)$:

$$\Delta\varphi = 2H(u_{0x} \wedge \varphi_y + \varphi_x \wedge u_{0y})\ .$$

It follows from a result of Wente [5; Lemma 3.1] that $\varphi \in L^\infty(\Omega;\mathbb{R}^3)$; see also Brezis-Coron [2; Lemma A.1] or Struwe [18; Theorem III.5.1]. Hence by minimality of u_0 in $u_0 + H_0^{1,2} \cap L^\infty(\Omega;\mathbb{R}^3)$, for small $|t|$ from Lemma 5.3 we have

$$\begin{aligned} E_H(u_0) \le E_H(u_0 + t\varphi) &= E_H(u_0) + t\langle\varphi, DE_H(u_0)\rangle \\ &\quad + \frac{t^2}{2}D^2E_H(u_0)(\varphi,\varphi) + 2Ht^3V(\varphi) \\ &= E_H(u_0) + 2Ht^3V(\varphi)\ , \end{aligned}$$

and it follows that $V(\varphi) = 0$; that is, $E_H(u_0 + t\varphi) = E_H(u_0)$ for all $t \in \mathbb{R}$. But then for small $|t|$ also $u_0 + t\varphi$ is a relative minimizer of E_H and satisfies (5.1), (5.2):

$$\Delta(u_0 + t\varphi) = 2H(u_0 + t\varphi)_x \wedge (u_0 + t\varphi)_y\ .$$

Since $H \neq 0$, differentiating twice with respect to t gives

$$0 = \varphi_x \wedge \varphi_y\ .$$

But then $D^2V(u_0)(\varphi,\varphi) = 2\int_\Omega u_0 \cdot \varphi_x \wedge \varphi_y\, dz = 0$, and we obtain

$$\delta = d^2E_H(u_0)(\varphi,\varphi) = 2E_0(\varphi) = 2\ ,$$

contrary to the assumption about δ. □

Proof of Lemma 5.6. Let (u_m) be a (P.-S.)$_\beta$-sequence in $u_0 + H_0^{1,2}(\Omega;\mathbb{R}^3)$. Consider $\varphi_m = u_m - u_0$ and note that by Lemma 5.3

$$\begin{aligned} E_H(u_m) = E_H(u_0 + \varphi_m) &= E_H(u_0) + \langle\varphi_m, DE_H(u_0)\rangle \\ &\quad + \frac{1}{2}D^2E_H(u_0)(\varphi_m,\varphi_m) + 2HV(\varphi_m) \\ &= E_H(u_0) + \frac{1}{2}D^2E_H(u_0)(\varphi_m,\varphi_m) + 2HV(\varphi_m)\ , \end{aligned}$$

while

$$\begin{aligned} o(1)\big(E_0(\varphi_m)\big)^{\frac{1}{2}} &= \langle \varphi_m, DE_H(u_m)\rangle \\ &= \langle \varphi_m, DE_H(u_0)\rangle + D^2E_H(u_0)(\varphi_m,\varphi_m) + 6HV(\varphi_m) \\ &= D^2E_H(u_0)(\varphi_m,\varphi_m) + 6HV(\varphi_m)\ , \end{aligned}$$

where $o(1) \to 0$ $(m \to \infty)$. Subtracting, we obtain

$$D^2E_H(u_0)(\varphi_m,\varphi_m) = 6\big(E_H(u_m) - E_H(u_0)\big) + o(1)\big(E_0(\varphi_m)\big)^{\frac{1}{2}}\ ,$$

and by Lemma 5.7 it follows that (φ_m) and hence (u_m) is bounded. We may assume that $u_m \rightharpoonup u$ weakly. Then for any $\varphi \in H_0^{1,2}(\Omega;\mathbb{R}^3)$ by Lemma 5.3 also

$$o(1) = \langle \varphi, DE_H(u_m)\rangle \to \langle \varphi, DE_H(u)\rangle\ ,$$

and it follows that u solves (5.1), (5.2). Thus, $t = 1$ is a critical point for the cubic function

$$t \mapsto E_H(u_0 + t(u - u_0))$$

which also attains a relative minimum at $t = 0$, and we conclude that

$$E_H(u) \geq E_H(u_0)\ .$$

Now let $\psi_m = u_m - u$ and note that by Lemma 5.3 we have

$$\begin{aligned} E_H(u_m) &= E_H(u) + E_0(\psi_m) + HD^2V(u)(\psi_m,\psi_m) + 2HV(\psi_m) \\ &= E_H(u) + E_H(\psi_m) + HD^2V(u)(\psi_m,\psi_m) \\ &= E_H(u) + E_H(\psi_m) + o(1)\ , \end{aligned}$$

$$\begin{aligned} o(1) &= \langle \psi_m, DE_H(u_m)\rangle = \langle \psi_m, DE_H(\psi_m)\rangle + 2HD^2V(u)(\psi_m,\psi_m) \\ &= \langle \psi_m, DE_H(\psi_m)\rangle + o(1)\ , \end{aligned}$$

with error $o(1) \to 0$ as $m \to \infty$. That is, for m sufficiently large

$$\begin{aligned} E_H(\psi_m) &= E_0(\psi_m) + 2HV(\psi_m) \leq E_H(u_m) - E_H(u) + o(1) \\ &\leq E_H(u_m) - E_H(u_0) + o(1) \leq c < \frac{4\pi}{3H^2}\ , \end{aligned}$$

while

$$\frac{1}{2}\langle \psi_m, DE_H(\psi_m)\rangle = E_0(\psi_m) + 3HV(\psi_m) = o(1)\ .$$

It follows that for $m \geq m_0$ there holds

$$E_0(\psi_m) \leq c < \frac{4\pi}{H^2}\ ,$$

whence by the isoperimetric inequality Theorem 5.4 we have

$$\begin{aligned} o(1) &= E_0(\psi_m) + 3HV(\psi_m) \\ &\geq E_0(\psi_m)\left(1 - \sqrt{\frac{H^2E_0(\psi_m)}{4\pi}}\right) \geq cE_0(\psi_m) \end{aligned}$$

for some $c > 0$, and $\psi_m \to 0$ strongly, as desired. □

Large Solutions

We can now state the main result of this section, the existence of "large" solutions to the Dirichlet problem (5.1), (5.2), due to Brezis-Coron [2] and Struwe [12], [11], with a contribution by Steffen [2].

5.8 Theorem. *Suppose $u_0 \not\equiv const.$, $H \neq 0$ and assume that E_H admits a relative minimum $\underline{u}$ on $u_0 + H_0^{1,2}(\Omega; \mathbb{R}^3)$. Then there exists a further solution $\overline{u}$, distinct from $\underline{u}$, of (5.1), (5.2).*

Proof. We may assume that $\underline{u} = u_0$. Then by Lemma 5.6 the theorem follows from Theorem II.6.1 once we establish the following two conditions:
(1°) There exists u_1 such that $E_H(u_1) < E_H(u_0)$.
(2°) Letting

$$P = \left\{ p \in C^0\big([0,1]; u_0 + H_0^{1,2}(\Omega; \mathbb{R}^3)\big);\, p(0) = u_0,\, p(1) = u_1 \right\} ,$$

we have

$$\beta = \inf_{p \in P} \sup_{u \in p} E_H(u) < E_H(u_0) + \frac{4\pi}{3H^2} .$$

Conditions (1°) and (2°) will be established by using the sphere-attaching mechanism of Wente [2; p. 285 f.], [3], refined by Brezis and Coron [2; Lemma 5]: Since $\underline{u} = u_0 \not\equiv const.$, there exists some point $z_0 = (x_0, y_0) \in \Omega$ such that $\nabla \underline{u}(z_0) \neq 0$. By conformal invariance of (5.1) we may assume that $z_0 = 0$, $\underline{u}(z_0) = 0$, and that with

$$\underline{u}_x(0) = a = (a_1, a_2, a_3),\ \underline{u}_y = b = (b_1, b_2, b_3)$$

there holds

$$H(a^1 + b^2) < 0 .$$

For $\varepsilon > 0$ now let $\varphi_\varepsilon(x, y) = \frac{2\varepsilon}{\varepsilon^2 + x^2 + y^2}(x, y, \varepsilon)$ be the stereographic projection of $\mathbb{R}^2$ onto a sphere of radius 1 centered at $(0, 0, 1)$ considered earlier, and let $\xi \in C_0^\infty(\Omega)$ be a symmetric cut-off function such that $\xi(z) = \xi(-z)$ and $\xi \equiv 1$ in a neighborhood of $z_0 = 0$. Define

$$u_t = u_t^\varepsilon := \underline{u} + t\xi\varphi_\varepsilon \in \underline{u} + H_0^{1,2}(\Omega; \mathbb{R}^3) .$$

For small $\varepsilon > 0$ the surface u_t "looks" like a sphere of radius t attached to $\underline{u}$ above $\underline{u}(0)$.

Now compute, using (5.4),

$$\begin{aligned} E_H(u_t) &= E_H(\underline{u}) + \frac{t^2}{2} D^2 E_H(\underline{u})(\xi\varphi_\varepsilon, \xi\varphi_\varepsilon) + 2Ht^3 V(\xi\varphi_\varepsilon) \\ &= E_H(\underline{u}) + t^2 E_0(\xi\varphi_\varepsilon) + 2Ht^3 V(\xi\varphi_\varepsilon) \\ &\quad + 2Ht^2 \int_\Omega \underline{u} \cdot (\xi\varphi_\varepsilon)_x \wedge (\xi\varphi_\varepsilon)_y \, dx\, dy . \end{aligned}$$

Clearly

$$E_0(\xi\varphi_\varepsilon) \le E_0(\varphi_\varepsilon; \mathbb{R}^2) + O(\varepsilon^2) = 4\pi + O(\varepsilon^2) \ ,$$

while

$$V(\xi\varphi_\varepsilon) = V(\varphi_\varepsilon; \mathbb{R}^2) + O(\varepsilon^3) = \frac{4\pi}{3} + O(\varepsilon^3) \ .$$

Expand

$$\underline{u}(x,y) = \underline{u}(0) + \underline{u}_x(0)x + \underline{u}_y(0)y + O(r^2) = ax + by + O(r^2) \ ,$$

where $r^2 = x^2 + y^2$. Upon integrating by parts and using anti-symmetry of the wedge product, we obtain the following expression:

$$\begin{aligned}
2H &\int_\Omega (ax+by)\cdot(\xi\varphi_\varepsilon)_x \wedge (\xi\varphi_\varepsilon)_y \, dx\, dy \\
&= H\int_\Omega \big(a\wedge(\xi\varphi_\varepsilon)_y + (\xi\varphi_\varepsilon)_x \wedge b\big)\cdot(\xi\varphi_\varepsilon)\, dx\, dy \\
&= H\int_\Omega \big(a\wedge(0,1,0) + (1,0,0)\wedge b\big)\cdot \xi^2 \frac{4\varepsilon^2}{(\varepsilon^2+r^2)^2}(x,y,\varepsilon)\, dx\, dy \\
&= 4H(a_1+b_2)\int_\Omega \frac{\varepsilon^3}{(\varepsilon^2+r^2)^2}\xi^2\, dx\, dy \\
&\qquad - 4H\int_\Omega (a_3x+b_3y)\frac{4\varepsilon^2}{(\varepsilon^2+r^2)^2}\xi^2\, dx\, dy \ .
\end{aligned}$$

Since ξ is symmetric the last term vanishes, while for sufficiently small $\varepsilon > 0$

$$\int_\Omega \frac{\varepsilon^3}{(\varepsilon^2+r^2)^2}\xi^2\, dx\, dy \ge \frac{1}{4\varepsilon}\int_{B_\varepsilon(0)} dx\, dy = \frac{\varepsilon\pi}{4} > 0 \ .$$

Finally, since $|\nabla\varphi_\varepsilon(x,y)| \le \frac{c\varepsilon}{\varepsilon^2+x^2+y^2}$, we may estimate

$$\begin{aligned}
&\left|\int_\Omega O(r^2)\cdot(\xi\varphi_\varepsilon)_x \wedge (\xi\varphi_\varepsilon)_y\, dx\, dy\right| \\
&\qquad \le \int_\Omega O(r^2)|\nabla\varphi_\varepsilon|^2\, dx\, dy + O(\varepsilon^2) \\
&\qquad \le c\int_\Omega \frac{\varepsilon^2 r^2}{(\varepsilon^2+r^2)^2}\, dx\, dy + O(\varepsilon^2) \\
&\qquad \le c\int_{B_\varepsilon(0)} dx\, dy + c\int_{\Omega\setminus B_\varepsilon(0)} \frac{\varepsilon^2}{r^2}\, dx\, dy + O(\varepsilon^2) \\
&\qquad \le c\varepsilon^2 + c\varepsilon^2|ln\ \varepsilon| \ .
\end{aligned}$$

Hence, for $t \ge 0$ we may estimate

$$E_H(u_t) \le E_H(\underline{u}) + t^2\big(4\pi + H(a_1+b_2)\pi\varepsilon + c\varepsilon^2|ln\ \varepsilon| + c\varepsilon^2\big) + 2Ht^3\big(\frac{4\pi}{3} + O(\varepsilon^3)\big) \ .$$

In particular, if $H < 0$ and if $\varepsilon > 0$ is chosen sufficiently small, there exists $T > 0$ such that u_T satisfies $E_H(u_T) < E_H(\underline{u})$ and, moreover,

$$\sup_{0 \le t \le T} E_H(u_t) < E_H(\underline{u}) + \frac{4\pi}{3H^2} \ ,$$

as desired. The case $H > 0$ follows in a similar way by considering $(u_t)_{t \le 0}$. □

5.9 Remarks. A "global" compactness condition analogous to Theorem 3.1 also holds in the case of Equation (5.1); see Brezis-Coron [3], Struwe [11]. As a consequence, it is not hard to see the analogue of Theorem 3.4 (Rentzmann [1], Patnaik [1]; if Ω is an annulus the existence of non-trivial solutions $u \in H_0^{1,2}(\Omega; \mathbb{R}^3)$ to (5.1) was already shown by Wente [4]). It would be interesting to further investigate the effect of topology, for instance, in the spirit of Bahri-Coron [1].

For further material and references on the Dirichlet and Plateau problems for surfaces of prescribed mean curvature we refer the interested reader to the lecture notes by Struwe [18]. A result analogous to Theorem 5.8 for variable mean curvature $H = H(u)$ has recently been obtained by Struwe [19]; see also Wang, G.F. [1].

6. Harmonic Maps of Riemannian Surfaces

As our final example we now present a borderline variational problem in a non-smooth setting, where critical points under suitable conditions can be obtained via a flow approach similar to Section 4.

Given a smooth, closed Riemannian surface Σ with metric γ and any smooth, closed k-manifold N with metric g, a natural generalization of Dirichlet's integral for C^1-functions on a domain in $\mathbb{R}^n$ is the energy functional

$$E(u) = \int_\Sigma e(u)\, d\mu \ .$$

Here, in local coordinates on Σ and N, the energy density $e(u)$ is given by

$$e(u) = \sum_{1 \le \alpha, \beta \le 2} \sum_{1 \le i,j \le k} \frac{1}{2} \gamma^{\alpha\beta}(x) g_{ij}(u) \frac{\partial}{\partial x_a} u^i \frac{\partial}{\partial x_\beta} u^j \ ,$$

with $\gamma^{\alpha\beta} = (\gamma_{\alpha\beta})^{-1}$ denoting the coefficients of the inverse of the matrix $(\gamma_{\alpha\beta})$ representing the metric γ, (g_{ij}) representing g, and with area element

$$d\mu = \sqrt{|\gamma|}\, dx, \ |\gamma| = \det(\gamma_{\alpha\beta}) \ .$$

Since we assume that both Σ and N are compact, this expression may be simplified considerably: First, by the Nash embedding theorem, see for instance

Nash [1] or Schwartz [2; pp. 43–53], any compact manifold N may be isometrically embedded into some Euclidean $\mathbb{R}^n$. A new proof – avoiding the "hard" implicit function theorem – has recently been obtained by Günther [1].

Moreover, E is invariant under conformal mappings of Σ. Thus, by the uniformization theorem, we may assume that either $\Sigma = S^2$ or $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, or is a quotient of the upper half-space $\mathbb{H}$, endowed with the hyperbolic metric $\frac{1}{y}dx\ dy$. In particular, if $\Sigma = T^2$, the energy density is simply given by $e(u) = \frac{1}{2}|\nabla u|^2$ and E becomes the standard Dirichlet integral for mappings $u\colon T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to N \subset \mathbb{R}^n$.

Consider the space $C^1(\Sigma; N)$ of C^1-functions $u\colon \Sigma \to N \hookrightarrow \mathbb{R}^n$. Note that $C^1(\Sigma; N)$ is a manifold with tangent space given by

$$T_u C^1(\Sigma; N) = \left\{\varphi \in C^1(\Sigma; \mathbb{R}^n)\ ;\ \varphi(x) \in T_{u(x)}N \ \text{ for } x \in \Sigma\right\}\ ,$$

and E is differentiable on this space.

In fact, if we consider variations $\varphi \in T_u C^1(\Sigma; N)$ such that $supp(\varphi)$ is contained in a chart U on Σ whose image $u(U)$ is contained in a coordinate patch of N, then in order to compute the variation of E in direction φ it suffices to work in *one* coordinate frame – both on Σ and N – and all computations can be done as in the "flat" case. From this, the differentiability of E is immediate and the following definition is meaningful.

6.1 Definition. *A stationary point $u \in C^1(\Sigma; N)$ of E is called a harmonic map.*

The concept of harmonic map generalizes the notion of (closed) geodesic to higher dimensions; compare Section II.4. Moreover, if we choose $N = \mathbb{R}^n$, we see that harmonic functions simply appear as special cases of harmonic maps. However, when the target space is curved, the Euler-Lagrange equations for harmonic maps no longer will be linear, as we shall presently see.

The Euler-Lagrange Equations for Harmonic Maps

First consider the model case $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, where E reduces to the standard Dirichlet integral for doubly periodic mappings $u\colon \mathbb{R}^2 \to N \subset \mathbb{R}^n$, restricted to a fundamental domain.

In this case, if $u\colon T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to N \subset \mathbb{R}^n$ is harmonic of class C^2, the first variation of E gives

$$0 = \langle \varphi, DE(u)\rangle = \int_{T^2} \nabla u \nabla \varphi\ dx = -\int_{T^2} \Delta u\ \varphi\ dx\ ,$$

for all doubly periodic $\varphi \in C^1(\mathbb{R}^2; \mathbb{R}^n)$ satisfying the condition

$$\varphi(x) \in T_{u(x)}N \text{ for all } x \in \mathbb{R}^2\ .$$

That is, $-\Delta u(x)$ is orthogonal to the tangent space of N at the point $u(x)$, for any $x \in T^2$; in symbols:

$$-\Delta u(x) = \sum_{i=k+1}^{n} \lambda_i(x)\nu_i(u(x)) \perp T_{u(x)}N \qquad \text{for all } x \in T^2 \ ,$$

where $\nu_{k+1}, \dots, \nu_n$ is a smooth local orthonormal frame for the normal bundle $TN^\perp$ near $u(x)$, and where $\lambda_{k+1}, \dots, \lambda_n$ are scalar functions. In general, the Laplace operator will be replaced by the Laplace-Beltrami operator Δ_Σ on Σ.

The particular structure of this equation can best be seen when $N = S^{n-1} \subset \mathbb{R}^n$. In this case, if $u: T^2 \to S^{n-1} \subset \mathbb{R}^n$ is of class C^2 and harmonic, it follows that

$$-\Delta u = \lambda u$$

for some continuous function $\lambda: T^2 \to \mathbb{R}$. Testing this relation with u and noting that $|u| \equiv 1$, $u \cdot \nabla u \equiv 0$, we see that

$$\lambda = -\Delta u \cdot u = -\operatorname{div}(u \cdot \nabla u) + |\nabla u|^2 = |\nabla u|^2 \ ;$$

that is, harmonic maps into spheres satisfy the relation

$$-\Delta u = u|\nabla u|^2 \ .$$

For general target manifolds N we may proceed similarly. Fix an index $i \in \{k+1, \dots, n\}$. Then, since $\partial_\alpha u = \frac{\partial}{\partial x_\alpha} u \in T_u N$ for any α, we have

$$\begin{aligned} \lambda_i &= -\langle \Delta u, \nu_i \circ u \rangle \\ &= -\operatorname{div}\langle \nabla u, \nu_i \circ u \rangle + \langle \nabla u, (d\nu_i \circ u) \cdot \nabla u \rangle = \langle \nabla u, (d\nu_i \circ u) \cdot \nabla u \rangle \, , \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. That is, we have

$$-\Delta u = A(u)(\nabla u, \nabla u) \, ,$$

where $A(u): T_u N \times T_u N \to T_u^\perp N$ denotes the second fundamental form of N, given by

$$A(p)(\xi, \eta) = \sum_{i=k+1}^{n} \nu_i(p) A_i(p)(\xi, \eta) \, , \quad A_i(p)(\xi, \eta) = \langle \xi, d\nu_i(p)\eta \rangle \ ,$$

for $p \in N$, $\xi, \eta \in T_p N$.

For general domains, similarly the harmonic map equation reads

$$-\Delta_\Sigma u = A(u)(\nabla u, \nabla u)_\Sigma \perp T_u N \, ,$$

with

$$A(u)(\nabla u, \nabla u)_\Sigma = \sum_{\alpha,\beta} \gamma^{\alpha\beta} A(u)(\partial_\alpha u, \partial_\beta u) \ .$$

Note that by compactness of N the coefficients of the form A are uniformly bounded.

Bochner Identity

Upon differentiating the harmonic map equation and taking the scalar product with the components of ∇u, we obtain the equation

$$\begin{aligned} -\Delta e(u) + |\nabla^2 u|^2 &= -\frac{1}{2}\Delta\big(|\nabla u|^2\big) + |\nabla^2 u|^2 \\ &= -\nabla(\Delta u)\cdot\nabla u = \nabla\big(A(u)(\nabla u,\nabla u)\big)\cdot\nabla u \\ &= \operatorname{div}\big(A(u)(\nabla u,\nabla u)\cdot\nabla u\big) - A(u)(\nabla u,\nabla u)\cdot\Delta u \\ &= |A(u)(\nabla u,\nabla u)|^2 \le Ce(u)^2 \end{aligned}$$

for the energy density $e(u)$. In the model case of harmonic maps $u\colon T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to S^{n-1} \hookrightarrow \mathbb{R}^n$, this identity simply takes the form

$$-\Delta e(u) + |\nabla^2 u|^2 = |\nabla u|^4 = 4e(u)^2 \ .$$

For a general domain manifold Σ, additional terms related to the curvature of Σ appear. If we do not care about the precise form of these terms we then obtain the inequality

$$-\Delta_\Sigma e(u) + |\nabla du|^2_\Sigma \le C(1+e(u))e(u)$$

with a constant C depending on N and Σ.

The most interesting variant of the Bochner identity results if we work intrinsically on the manifold N and use covariant differentiation instead of taking the ordinary gradient. Then we obtain the differential inequality

$$-\Delta_\Sigma e(u) \le \kappa_N e(u)^2 + Ce(u)$$

for the energy density of u, where $\kappa_N \ge 0$ denotes an upper bound for the sectional curvature of N, and where C denotes a constant depending only on Σ and N. See for instance Jost [3], formula (3.2.10).

With these remarks we hope that the reader has become somewhat familiar with the concept of harmonic maps. Moreover, to fix ideas, in the following one may always think of mappings $u\colon T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to S^{n-1} \subset \mathbb{R}^n$. In this special case already, all essential difficulties appear and nothing of the flavour of the results will be lost.

The Homotopy Problem and its Functional Analytic Setting

A natural generalization of Dirichlet's problem for harmonic functions is the following "Homotopy Problem": Given a map $u_0\colon \Sigma \to N$, is there a harmonic map u homotopic to u_0?

As in the case of a scalar function $u\colon \Omega \to \mathbb{R}$ we may attempt to approach this problem by direct methods. Denote

$$H^{1,2}(\Sigma;N) = \big\{u \in H^{1,2}(\Sigma;\mathbb{R}^n)\ ;\ u(x) \in N \ \text{ for a.e. } x \in \Sigma\big\}$$

the space of $H^{1,2}$-mappings into N. (If $\Sigma = T^2$, then $H^{1,2}(\Sigma;N)$ is the space of mappings $u \in H^{1,2}_{loc}(\mathbb{R}^2;N)$ of period 1 in both variables, restricted to a fundamental domain.) Note that E is weakly lower semi-continuous and coercive on $H^{1,2}(\Sigma;N)$ with respect to $H^{1,2}(\Sigma;\mathbb{R}^n)$.

Moreover, by a result of Schoen-Uhlenbeck [2; Section IV] we have:

6.2 Theorem. *(1°) The space $C^\infty(\Sigma;N)$ of smooth maps $u\colon \Sigma \to N \subset \mathbb{R}^n$ is dense in $H^{1,2}(\Sigma;N)$.*
(2°) The homotopy class of any map $u_0 \in H^{1,2}(\Sigma;N)$ is well-defined.

Proof. (1°) The following argument basically is due to Courant [1; p. 214 f.]. Let $\varphi \in C^\infty_0\big(B_1(0)\big)$ satisfy $0 \le \varphi \le 1$, $\int_{B_1(0)} \varphi\ dx = 1$, and for $\varepsilon > 0$ let $\varphi_\varepsilon = \varepsilon^{-2}\varphi\left(\frac{x}{\varepsilon}\right)$. Given $u \in H^{1,2}(\Sigma;\mathbb{R}^n)$, denote

$$u * \varphi_\varepsilon(x_0) = \int_{B_\varepsilon(x_0)} u(x)\varphi_\varepsilon(x_0 - x)\ dx$$

its mollification with φ_ε (in local coordinates on Σ). Note that $u * \varphi_\varepsilon \in C^\infty(\Sigma;\mathbb{R}^n)$ and $u * \varphi_\varepsilon \to u$ in $H^{1,2}(\Sigma;\mathbb{R}^n)$ as $\varepsilon \to 0$. Let $\fint$ denote average and let $\operatorname{dist}(P,N) = \inf\{|P - Q|\ ;\ Q \in N\}$ denote the distance of a point P from N.

For $x_0 \in \Sigma, \varepsilon > 0$ and $y \in B_\varepsilon(x_0)$ now estimate

$$\begin{aligned}\operatorname{dist}^2\big(u * \varphi_\varepsilon(x_0), N\big) &\le \left|\int_{B_\varepsilon(x_0)} \big(u(x) - u(y)\big)\varphi_\varepsilon(x_0 - x)\ dx\right|^2 \\ &\le C \fint_{B_\varepsilon(x_0)} \big|u(x) - u(y)\big|^2\ dx\ .\end{aligned}$$

Taking the average with respect to $y \in B_\varepsilon(x_0)$, therefore we can bound

$$\begin{aligned}\operatorname{dist}^2\big(u * \varphi_\varepsilon(x_0), N\big) &\le C \fint_{B_\varepsilon(x_0)} \fint_{B_\varepsilon(x_0)} |u(x) - u(y)|^2\ dx\ dy \\ &\le C\varepsilon^2 \fint_{B_\varepsilon(x_0)} \fint_{B_\varepsilon(x_0)} \int_0^1 \big|\nabla u\big(y + t(x-y)\big)\big|^2\ dt\ dx\ dy \\ &\le C \int_{B_\varepsilon(x_0)} |\nabla u|^2\ dx\,.\end{aligned}$$

By absolute continuity of the Lebesgue integral, the latter is small for small ε, uniformly in x_0. In particular, for $\varepsilon < \varepsilon_0 = \varepsilon_0(u,N)$ the distance of $u_\varepsilon = u * \varphi_\varepsilon$ from N is smaller than $d/2$, where $d = d(N) > 0$ is the focal distance from N. We then can smoothly project u_ε down to N to obtain a function $\tilde{u}_\varepsilon \in C^\infty(\Sigma;N)$, satisfying

$$(u_\varepsilon - \tilde{u}_\varepsilon) \perp T_{\tilde{u}_\varepsilon}N\ ,\ \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^\infty} \to 0 \text{ as } \varepsilon \to 0\ .$$

Since $u_\varepsilon \to u$ in $H^{1,2}$ as $\varepsilon \to 0$,

$$\|\tilde{u}_\varepsilon - u\|_{L^2} \le \|\tilde{u}_\varepsilon - u_\varepsilon\|_{L^2} + \|u_\varepsilon - u\|_{L^2} \to 0 \text{ as } \varepsilon \to 0 \ .$$

Moreover, representing u_ε locally as $u_\varepsilon = \tilde{u}_\varepsilon + \sum_i \lambda_{i\varepsilon}\nu_i \circ \tilde{u}_\varepsilon$, by orthogonality we have

$$|\nabla u_\varepsilon|^2 \ge |\nabla \tilde{u}_\varepsilon + \sum_i \lambda_{i\varepsilon} \nabla(\nu_i \circ \tilde{u}_\varepsilon)|^2 \ge \big(1 - C\sum_i \|\lambda_{i\varepsilon}\|_{L^\infty}\big)|\nabla \tilde{u}_\varepsilon|^2 \ .$$

Hence

$$\limsup_{\varepsilon\to 0} \|\nabla \tilde{u}_\varepsilon\|_{L^2} \le \lim_{\varepsilon \to 0} \|\nabla u_\varepsilon\|_{L^2} = \|\nabla u\|_{L^2} \ ,$$

and it follows that $\tilde{u}_\varepsilon \to u$ in $H^{1,2}(\Sigma; N)$, as desired.

(2°) Define the homotopy class of $u \in H^{1,2}(\Sigma; N)$ as the homotopy class of any map $v \in C^\infty(\Sigma; N)$ with $\|v - u\|_{H^{1,2}} \le \delta$ for some $\delta = \delta(u, N) > 0$ to be determined. To see that this is well-defined, let $v_0,\ v_1 \in C^\infty(\Sigma; N)$ satisfy $\|v_i - u\|_{H^{1,2}} \le \delta$, $i = 0, 1$, and let $v_t = (1-t)v_0 + tv_1$ be a homotopy connecting v_0 and v_1 in $C^\infty(\Sigma; \mathbb{R}^n)$. Note that

$$\|v_t - u\|_{H^{1,2}} \le (1-t)\|v_0 - u\|_{H^{1,2}} + t\|v_1 - u\|_{H^{1,2}} \le \delta$$

for $0 \le t \le 1$. Moreover, for any $w \in H^{1,2}$, $\varepsilon > 0$ we can estimate

$$\|w * \varphi_\varepsilon\|_{L^\infty} \le C\|w * \varphi_\varepsilon\|_{H^{2,2}} \le C(1 + \varepsilon^{-1})\|w\|_{H^{1,2}} \ .$$

Thus for $0 < \varepsilon_0 = \varepsilon_0(u, N) \le 1$ as determined in (1°) and $0 \le t \le 1$ we have

$$\begin{aligned} \operatorname{dist}(v_t * \varphi_{\varepsilon_0}, N) &\le \operatorname{dist}(u * \varphi_{\varepsilon_0}, N) + \|(v_t - u) * \varphi_{\varepsilon_0}\|_{L^\infty} \\ &\le \operatorname{dist}(u * \varphi_{\varepsilon_0}, N) + C_1\varepsilon_0^{-1}\|v_t - u\|_{H^{1,2}} \le d/2 + C_1\delta/\varepsilon_0 \le 3d/4 \,, \end{aligned}$$

provided that we choose $4C_1\delta \le d\varepsilon_0$. We may then project the maps $v_t * \varphi_{\varepsilon_0}$ to maps $v_t \widetilde{*\varphi}_{\varepsilon_0} \in C^\infty(\Sigma; N)$, $0 \le t \le 1$, connecting $v_0 \widetilde{*\varphi}_{\varepsilon_0}$ and $v_1 \widetilde{*\varphi}_{\varepsilon_0}$.

For $i = 0, 1$ and $0 < \varepsilon < \varepsilon_0(u, N)$, finally, as in (1°) we can estimate

$$\operatorname{dist}^2(v_i * \varphi_\varepsilon(x_0), N) \le C\int_{B_\varepsilon(x_0)} |\nabla v_i|^2 \, dx \le C\int_{B_\varepsilon(x_0)} |\nabla u|^2 \, dx + C\delta \le 3d/4 \ ,$$

provided $\delta = \delta(u, N) > 0$ is chosen sufficiently small. Thus v_0 and $v_0 \widetilde{*\varphi}_{\varepsilon_0}$ are homotopic through $v_0 \widetilde{*\varphi}_\varepsilon$, $0 \le \varepsilon \le \varepsilon_0$, and similarly for v_1, showing that v_0 and v_1 belong to the same homotopy class. □

Bethuel [1] showed that on an m-ball B for any $m > 2$ the space $C^\infty(B; N)$ is dense in $H^{1,2}(B; N)$ if and only if $\pi_2(N) = 0$. For general domain manifolds, Bethuel's work was completed by Hang-Lin [1], after they uncovered a mistake in Bethuel's original work dealing with this case.

Motivated by Theorem 6.2 one could attempt to solve the homotopy problem by minimizing E in the homotopy class of u_0. However, while $H^{1,2}(\Sigma; N)$ is

weakly closed in the topology of $H^{1,2}(\Sigma;\mathbb{R}^n)$, in general this will not be the case for homotopy classes of non-constant maps. Consider for example the family $(\varphi_\varepsilon)_{\varepsilon>0}$ of stereographic projections to the standard sphere, introduced in (5.5); projecting back with φ_1, we obtain a family of maps $u_\varepsilon = \varphi_\varepsilon \circ (\varphi_1)^{-1}: S^2 \to S^2$ of degree 1, converging weakly to a constant map. Therefore, the direct method fails to be applicable for solving the homotopy problem.

Also note that the space $H^{1,2}(\Sigma;N)$ is not a manifold; therefore the standard deformation lemma Theorem II.3.4 or II.3.11 cannot be applied. (An interesting variant was recently proposed by Duc [2].)

Existence and Non-existence Results

In fact, the infinum of E in a given homotopy class in general need not be attained. (See White [1] for further results in this regard.) Even worse, the homotopy problem may not always have a solution. The following result, analogous to the results of Pohožaev (Theorem 1.3) and Wente (Theorem 5.1) in our previous examples, is due to Eells-Wood [1].

6.3 Theorem. *Any harmonic map $u \in C^1(T^2;S^2)$ necessarily has topological degree $\neq 1$.*

In particular, there is no harmonic map homotopic to a map $u_0: T^2 \to S^2$ of degree $+1$.

This result shows that we may encounter some lack of compactness in attempting to find critical points of E. However, compactness can be restored under certain conditions. In a pioneering paper Eells-Sampson [1] have obtained the following result.

6.4 Theorem. *Suppose the sectional curvature κ_N of N is non-positive. Then for any map $u_0: \Sigma \to N$ there exists a harmonic map homotopic to u_0.*

Surprisingly, also a topological condition on the target may suffice to solve the homotopy problem. The following result was obtained independently by Lemaire [1] and Sacks-Uhlenbeck [1].

6.5 Theorem. *If $\pi_2(N) = 0$, then for any $u_0 \in H^{1,2}(\Sigma;N)$ there is a smooth harmonic map homotopic to u_0.*

For the proof of Theorem 6.4 Eells and Sampson consider an evolution problem similar to the flow approach that we have encountered in Section 4.

In fact, also Theorem 6.5 can be obtained from an in-depth analysis of the Eells-Sampson flow. Moreover, as we shall see, this analysis reveals a deep analogy between the harmonic map problem and the problems (1.1), (1.3), the Yamabe problem, and the Dirichlet problem for the equation of constant mean curvature (5.1).

The Heat Flow for Harmonic Maps

Given a map $u_0 \in C^\infty(\Sigma; N)$, Eells and Sampson propose to consider a solution $u\colon M \times [0,\infty[\to N \hookrightarrow \mathbb{R}^n$ of the initial value problem

$$u_t - \Delta_\Sigma u = A(u)(\nabla u, \nabla u)_\Sigma \perp T_u N \text{ in } \Sigma \times \mathbb{R}_+ , \tag{6.1}$$

$$u|_{t=0} = u_0 . \tag{6.2}$$

We write $u_t = \partial_t u$ for brevity. Equation (6.1) may be interpreted as the "L^2-gradient flow" for E; in particular, we have the energy inequality

$$\int_0^T \int_\Sigma |u_t|^2 \, d\mu \, dt + E\big(u(T)\big) \le E(u_0) ,$$

for all $T > 0$; see Lemma 6.8 below. Moreover, like the standard linear heat equation, the flow defined by (6.1) has certain smoothing properties. For example, analogous to the stationary (time-independent) case, for (6.1) there holds the Bochner-type inequality

$$(\partial_t - \Delta_\Sigma) e(u) + |\nabla du|^2_\Sigma \le C(1 + e(u)) e(u) .$$

Working intrinsically, as in Jost [3], formula (3.2.11), the bound for the leading term on the right can be improved and we obtain the differential inequality

$$(\partial_t - \Delta_\Sigma) e(u) \le \kappa_N e(u)^2 + C e(u) , \tag{6.3}$$

where $\kappa_N \ge 0$ again denotes an upper bound for the sectional curvature of N.

Proof of Theorem 6.4. If $\kappa_N \le 0$, estimate (6.3) implies a linear differential inequality for the energy density, and we obtain the existence of a global solution $u \in C^2(\Sigma \times \mathbb{R}_+, N)$ to the evolution problem (6.1), (6.2). Moreover, by the weak Harnack inequality for sub-solutions of parabolic equations (see Moser [2; Theorem 3]) and the energy inequality, the maximum of $|\nabla u|$ may be a-priori bounded in terms of the initial data. Again by the energy inequality, we can find a sequence of times $t_m \to \infty$ such that $u_t(t_m) \to 0$ in L^2 as $m \to \infty$, and a subsequence $(u(t_m))$ converges to a harmonic map homotopic to u_0. □

By Theorem 6.3, in general we cannot expect that (6.1), (6.2) admits a global smooth solution, converging asymptotically to a harmonic map. However, we propose to establish that (6.1), (6.2) always admits a global weak solution which is unique in its class, for arbitrary initial data $u \in H^{1,2}(\Sigma; N)$ and without imposing any topological or geometric restrictions on the target manifold. Moreover, space-time singularities of the flow or failure of $u(t)$ to converge as $t \to \infty$ may be uniquely associated with the "bubbling off" of harmonic spheres, as we will now make precise.

The Global Existence Result

Let $\exp_x: T_x\Sigma \to \Sigma$ denote the exponential map at a point $x \in \Sigma$. (If $\Sigma = T^2$, then $\exp_x(y) = x + y$.) The following result is due to Struwe [10].

6.6 Theorem. *For any $u_0 \in H^{1,2}(\Sigma; N)$ there exists a distribution solution $u: \Sigma \times \mathbb{R}_+ \to N$ of (6.1) which is smooth on $\Sigma \times \mathbb{R}_+$ away from at most finitely many points $(\bar{x}_k, \bar{t}_k)$, $0 < \bar{t}_k < \infty$, $1 \le k \le \bar{K}$, which satisfies the energy inequality $E(u(t)) \le E(u(s))$ for all $0 \le s \le t$, and which assumes its initial data continuously in $H^{1,2}(\Sigma; N)$. The solution u is unique in this class.*

At a singularity $(\bar{x}, \bar{t})$ a smooth harmonic map $\bar{u}: S^2 \to N$ separates in the sense that for sequences $x_m \to \bar{x}$, $t_m \nearrow \bar{t}$, $R_m \searrow 0$ as $m \to \infty$ the family

$$u_m(x) \equiv u\big(\exp_{x_m}(R_m x), t_m\big) \to \tilde{u} \qquad \text{in } H^{2,2}_{loc}(\mathbb{R}^2; N) \ ,$$

where $\tilde{u}: \mathbb{R}^2 \to N$ is smooth and harmonic with finite energy. By composition with stereographic projection, $\tilde{u}$ induces a non-constant, smooth harmonic map $\bar{u}: S^2 \cong \overline{\mathbb{R}^2} \to N$. See Fig. 6.1 for illustration.

As $t_m \to \infty$ suitably, the sequence of maps $u(t_m)$ converges weakly in $H^{1,2}(\Sigma; N)$ to a smooth harmonic map $u_\infty: \Sigma \to N$, and smoothly away from finitely many points $\tilde{x}_k$, $1 \le k \le \tilde{K}$, where again harmonic spheres separate in the sense described above. Moreover, we have

$$E(u_\infty) \le E(u_0) - K\,\varepsilon_0 \ ,$$

where $K = \bar{K} + \tilde{K}$ is the total number of singularities and where

$$\varepsilon_0 = \inf\big\{E(u) \ ; \ u \in C^1(S^2; N) \ \text{ is non-constant and harmonic } \big\} > 0$$

is a constant depending only on the geometry of N. In particular, the number of singularities of u is a-priori bounded, $K \le \varepsilon_0^{-1} E(u_0)$.

Theorem 6.6 implies Theorem 6.5:

Proof of Theorem 6.5. We follow Struwe [15; p. 299 f.]. Let $[u_0]$ be a homotopy class of maps from Σ into N. We may suppose that $[u_0]$ is represented by a smooth map $u_0: \Sigma \to N$ such that

$$E(u_0) \le \inf_{u \in [u_0]} E(u) + \frac{\varepsilon_0}{2} \ .$$

Let $u: \Sigma \times \mathbb{R}_+ \to N$ be the solution to the evolution problem (6.1), (6.2) constructed in Theorem 6.6. Suppose u first becomes singular or concentrates at a point $(\bar{x}, \bar{t})$, $\bar{t} \le \infty$. Then for sequences $t_m \nearrow \bar{t}$, $x_m \to \bar{x}$, $r_m \searrow 0$ we have

$$u_m(x) := u\big(\exp_{x_m}(r_m x), t_m\big) \to \tilde{u} \qquad \text{in } H^{2,2}_{loc}(\mathbb{R}^2; N)$$

where $\tilde{u}$ may be extended to a smooth harmonic map $\bar{u}: S^2 \to N$.

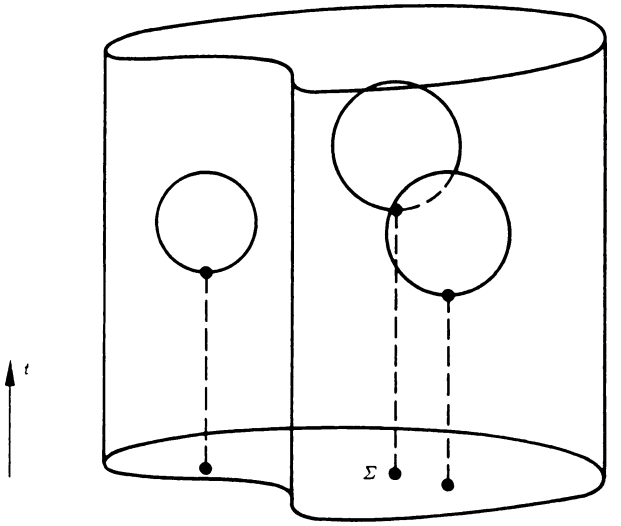

Fig. 6.1. "Separation of spheres"

In the image covered by u_m we now replace a large part of the "harmonic sphere" $\bar{u}$ by its "small" complement, "saving" at least $\varepsilon_0/2$ in energy. If $\pi_2(N) = 0$, this change of u_m will not affect the homotopy class $[u_m] = [u_0]$, and a contradiction will result.

More precisely, since $u_m \to \tilde{u}$ in $H^{2,2}_{loc}$ and since $E(\tilde{u}) < \infty$, we can find a sequence of radii $R_m \to \infty$ such that as $m \to \infty$ we have $r_m R_m \to 0$ while

$$\sup_{\partial B_{R_m}} |u_m - \tilde{u}| + R_m \int_{\partial B_{R_m}} |\nabla(u_m - \tilde{u})|^2 \, do \to 0 \, ,$$

$$\int_{\mathbb{R}^2 \setminus B_{R_m}} |\nabla \tilde{u}|^2 \, dx \to 0 \, ,$$

$$\int_{B_{R_m}} |\nabla(u_m - \tilde{u})|^2 \, dx \to 0 \, ,$$

where $B_{R_m} = B_{R_m}(0)$. Now let $\psi \in H^{1,2}(B_{R_m}; \mathbb{R}^n)$ solve the Dirichlet problem

$$-\Delta \psi_m = 0 \text{ in } B_{R_m}$$

with boundary data

$$\psi_m = u_m - \tilde{u} \qquad \text{on } \partial B_{R_m} \, .$$

By the maximum principle

$$\sup_{B_{R_m}} |\psi_m| = \sup_{\partial B_{R_m}} |u_m - \tilde{u}| \to 0 \, .$$

Moreover, by classical potential estimates and interpolation (see Lions-Magenes [1; Theorem 8.2]), the Dirichlet integral of ψ_m may be estimated in terms of the semi-norm of the trace $\psi_m|_{\partial B_{R_m}} \in H^{1/2,2}(\partial B_{R_m})$ and as $m \to \infty$ we obtain

$$\begin{aligned}\int_{B_{R_m}} |\nabla\psi_m|^2 \, dx &\le c|\psi_m|^2_{H^{1/2,2}(\partial B_{R_m})} \\ &\le c\int_{\partial B_{R_m}}\int_{\partial B_{R_m}} \frac{|\psi_m(x)-\psi_m(y)|^2}{|x-y|^2}\, dx\, dy \\ &\le c\, R_m \int_{\partial B_{R_m}} |\nabla'\psi_m|^2 \, do \le c\, R_m \int_{\partial B_{R_m}} |\nabla(u_m-\tilde u)|^2 \, do \to 0 \ ,\end{aligned}$$

where ∇' denotes the tangential gradient. (See for instance Adams [1; Theorem 7.48] for the equivalent integral representation of the $H^{1/2,2}$-semi-norm.)

Hence if we replace $u(t_m)$ by the map v_m, where

$$v_m\big(\exp_{x_m}(r_m x)\big) = \begin{cases} u_m(x), & |x| \ge R_m \\ \tilde u\big(R_m^2 \dfrac{x}{|x|^2}\big) + \psi_m(x), & |x| < R_m \ , \end{cases}$$

is defined via the exponential map in a small coordinate chart $\mathcal{U}$ around x_m, and where $v_m \equiv u(t_m)$ on $\Sigma \setminus \mathcal{U}$, we obtain that, as $m \to \infty$,

$$\operatorname{dist}\big(v_m(x), N\big) \le \sup_{B_{R_m}} |\psi_m| \to 0 \ ,$$

uniformly in $x \in \Sigma$. But then we may project v_m onto N to obtain a map $w_m \in H^{1,2}(\Sigma; N)$ satisfying, with error $o(1) \to 0$ as $m \to \infty$,

$$\begin{aligned} E(w_m) &\le E(v_m) + o(1) \\ &\le E(u_m) - \frac{1}{2}\int_{B_{R_m}} |\nabla u_m|^2 \, dx + \int_{\mathbb{R}^2\setminus B_{R_m}} |\nabla\tilde u|^2 \, dx \\ &\quad + \int_{B_{R_m}} |\nabla\psi_m|^2 \, dx + o(1) \\ &\le E(u_0) - \frac{1}{2}\int_{B_{R_m}} |\nabla\tilde u|^2 \, dx + o(1) \\ &\le \inf_{u\in[u_0]} E(u) + \varepsilon_0/2 - \varepsilon_0 + o(1) \end{aligned}$$

and the latter is strictly smaller than $\inf_{u\in[u_0]} E(u)$ for large m.

On the other hand, since $\pi_2(N) = 0$, the maps $x \mapsto \tilde u(x)$ and $x \mapsto \tilde u(R_m^2 x/|x|^2)$ for any m are homotopic as maps from B_{R_m} into N with fixed boundary; hence w_m is homotopic to $u(t_m)$, and the latter is homotopic to u_0 via the flow $(u(t))_{0\le t\le t_m}$, which is regular for all m by assumption. Thus, if the flow $(u(t))_{t>0}$ were to develop a singularity at some finite time $\bar t$ or a concentration as $t \to \infty$, there results a contradiction to our choice of u_0. Hence, the flow $(u(t))_{t>0}$ exists and is regular for all time and, as $t_m \to \infty$ suitably,

$u(t_m) \to u_\infty$ strongly in $H^{1,2}(\Sigma; N)$ where u_∞ is harmonic. But by Theorem 6.2, for sufficiently large m the maps u_∞ and $u(t_m)$ are homotopic, and the latter is homotopic to u_0 via the flow $(u(t))_{0\le t\le t_m}$. □

The Proof of Theorem 6.6

The following Sobolev interpolation estimate independently due to Gagliardi-Nirenberg and Ladyzhenskaya [1] plays a key role in the proof of Theorem 6.6.

6.7 Lemma. *For any function $u \in H^{1,2}_{loc}(\mathbb{R}^2)$, any $R > 0$, and any function $\varphi \in C^\infty_0(B_R)$ with $0 \le \varphi \le 1$ and such that $|\nabla\varphi| \le 4/R$, there holds*

$$\int_{\mathbb{R}^2} |u|^4\varphi^2 \, dx \le c_0 \left(\int_{B_R} |u|^2 dx\right) \cdot \left(\int_{B_R} |\nabla u|^2\varphi^2 \, dx + R^{-2}\int_{B_R} |u|^2 dx\right),$$

with c_0 independent of u and R.

Proof. The function $|u|^2\varphi$ has compact support. Thus, for any $x = (\xi, \eta) \in \mathbb{R}^2$ we have

$$\begin{aligned}\big|\big(|u|^2\varphi\big)(\xi,\eta)\big| &= \left|\int_{-\infty}^{\xi} \frac{\partial}{\partial\xi}\big(|u|^2\varphi\big)(\xi',\eta)\, d\xi'\right| \\ &\le 2\Big(\int_{-\infty}^{\infty} \big(|\nabla u|\,|u|\varphi + |u|^2|\nabla\varphi|\big)(\xi',\eta)\, d\xi' \,,\end{aligned}$$

and an analogous estimate with integration in η-direction.

Hence by Fubini's theorem and Hölder's inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} |u|^4\varphi^2(x)\, dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^4\varphi^2(\xi,\eta)\, d\xi\, d\eta \\ &\le 4\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\left(\int_{-\infty}^{\infty} \big|\nabla u|\,|u|\varphi + |u|^2\,|\nabla\varphi|\big)(\xi',\eta)\, d\xi'\right)\cdot \right. \\ &\qquad \left. \cdot \left(\int_{-\infty}^{\infty} \big(|\nabla u|\,|u|\varphi + |u|^2\,|\nabla\varphi|\big)(\xi,\eta')\, d\eta'\right)\right\} d\xi\, d\eta \\ &= 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(|\nabla u|\,|u|\varphi + |u|^2|\nabla\varphi|\big)(\xi',\eta)\, d\xi'\, d\eta\cdot \\ &\qquad \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(|\nabla u|\,|u|\varphi + |u|^2|\nabla\varphi|\big)(\xi,\eta') d\xi\, d\eta' \\ &= 4\left(\int_{\mathbb{R}^2} \big(|\nabla u|\,|u|\varphi + |u|^2|\nabla\varphi|\big)\, dx\right)^2 \\ &\le 8\left(\int_{\operatorname{supp}\varphi} |u|^2\, dx\right)\cdot \int_{\mathbb{R}^2} \big(|\nabla u|^2\varphi^2 + |u|^2|\nabla\varphi|^2\big)\, dx\,.\end{aligned}$$

Since $supp\,\varphi \subset B_R$ and $|\nabla\varphi| \le 4/R$, the claim follows. □

To see how we may apply Lemma 6.7, observe that by compactness of N with a uniform constant $C = C(N)$ for any solution u of (6.1) there holds

$$|u_t - \Delta_\Sigma u| \le C|\nabla u|^2 \,. \tag{6.4}$$

Hence, as shown in Lemmas 6.10 and 6.11 below, if the energy of $u(t)$ is uniformly small on small balls, from Lemma 6.7 applied to the function $|\nabla u|$ we obtain an L^2-estimate for $|\nabla^2 u|$, and then also higher regularity.

It is then of crucial importance to obtain energy bounds for $u(t)$, both globally and locally. First we note the following energy inequality.

6.8 Lemma. *Let $u \in C^2\big(\Sigma \times [0,T[; N\big)$ solve (6.1), (6.2). Then there holds*

$$\int_0^t \int_\Sigma |u_t|^2 \, d\mu \, dt' + E\big(u(t)\big) \le E(u_0) \text{ for all } t < T \,.$$

Proof. Multiply (6.1) by $u_t \in T_u N$ and use orthogonality $A(u)(\cdot,\cdot) \perp T_u N$ to obtain the identity

$$\langle u_t - \Delta_\Sigma u, u_t\rangle = |u_t|^2 + \frac{d}{dt} e(u) - \operatorname{div}\big(\langle \nabla u, u_t\rangle\big) = 0 \,. \tag{6.5}$$

The claim follows upon integrating over $\Sigma \times [0,t]$. □

Lemma 6.7 also calls for control of the energy density, locally. For this let $\imath_\Sigma$ be the injectivity radius of the exponential map on Σ. Then on any ball $B_R(x_0) \subset \Sigma$ of radius $R < \imath_\Sigma$ we may introduce Euclidean coordinates by a conformal change of variables. Recall that E is conformally invariant. With reference to such a conformal chart then introduce the local energy

$$E\big(u; B_R(x_0)\big) = \frac{1}{2} \int_{B_R(x_0)} |\nabla u|^2 \, dx \,, \quad R < \imath_\Sigma \,.$$

In the following we shift $x_0 = 0$ and again let $B_R = B_R(0)$ for brevity.

6.9 Lemma. *There is an absolute constant c_1 such that for any $R < \frac{1}{2}\imath_\Sigma$ and any solution $u \in C^2(B_{2R} \times [0,T]; N)$ to (6.1) with $\sup_{0\le t\le T} E\big(u(t); B_{2R}\big) \le E_0$ there holds*

$$E\big(u(T); B_R\big) \le E(u_0; B_{2R}) + c_1 \frac{T}{R^2} E_0 \,.$$

Proof. Consider $\Sigma = T^2$, for simplicity. Choose $\varphi \in C_0^\infty(B_{2R})$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in B_R and $|\nabla\varphi| \le 2/R$. Multiplying (6.5) by φ^2 and integrating by parts, we obtain

$$\begin{aligned} \int_0^T \int_{B_{2R}} \Big(|u_t|^2\varphi^2 + \frac{1}{2}\frac{d}{dt}\big(|\nabla u|^2\varphi^2\big)\Big) \, dx \, dt &= -2\int_0^T \int_{B_{2R}} \nabla u u_t \nabla\varphi\varphi \, dx \, dt \\ &\le \int_0^T \int_{B_{2R}} |u_t|^2\varphi^2 \, dx \, dt + \int_0^T \int_{B_{2R}} |\nabla u|^2 |\nabla\varphi|^2 \, dx \, dt \,. \end{aligned}$$

Therefore we can estimate

$$E\big(u(T);B_R\big) - E(u_0,B_{2R}) \le \frac{1}{2}\int_{B_{2R}} \big(|\nabla u|^2\varphi^2\big)\,dx\Big|_{t=0}^{T}$$
$$\le 4R^{-2}\int_0^T\int_{B_{2R}} |\nabla u|^2\,dx\,dt \le 8\frac{T}{R^2}E_0\ ,$$

as claimed. □

If the energy of a solution u to (6.1) is uniformly small on small balls, we can bound its second derivatives in terms of the energy.

6.10 Lemma. *There exists $\varepsilon_1 = \varepsilon_1(\Sigma,N) > 0$ with the following property: If $u \in C^2\big(B_{2R}\times[0,T[;N\big)$ solves (6.1) on $B_{2R}\times[0,T[$ for some $R \in]0,\frac{1}{2}\imath_\Sigma[$, and if*

$$\sup_{0\le t\le T} E\big(u(t),B_{2R}\big) < \varepsilon_1 \le E_0\ ,$$

then we have

$$\int_0^T\int_{B_R} |\nabla^2 u|^2\,dx\,dt \le c\,E_0\left(1+\frac{T}{R^2}\right)\ ,$$

where c depends only on Σ and N.

Proof. Let $Q = B_{2R}\times[0,T[$. Choose $\varphi \in C_0^\infty(B_{2R})$ such that $0\le\varphi\le 1,\ \varphi\equiv 1$ on $B_R(x)$, $|\nabla\varphi| \le 2/R$ and multiply (6.1) by $\Delta u\varphi^2$. Then from (6.4) we obtain that

$$\int_Q \left(\partial_t\big(\frac{|\nabla u|^2\varphi^2}{2}\big) + |\Delta u|^2\varphi^2\right)\,dx\,dt$$
$$\le C\int_Q |\nabla u|^2|\Delta u|\varphi^2\,dx\,dt - 2\int_Q u_t\nabla u\nabla\varphi\varphi\,dx\,dt$$
$$\le C\int_Q \big(|\nabla u|^4\varphi^2 + |\nabla u|^2|\nabla\varphi|^2\big)\,dx\,dt + \frac{1}{2}\int_Q |\Delta u|^2\varphi^2\,dx\,dt$$
$$\le C_1\varepsilon_1\int_Q |\nabla^2 u|^2\varphi^2\,dx\,dt + \frac{1}{2}\int_Q |\Delta u|^2\varphi^2\,dx\,dt$$
$$+ CR^{-2}\int_Q |\nabla u|^2\,dx\,dt\,.$$

Here we have used the fact that $u_t\nabla u = \Delta u\nabla u$ by (6.1). Young's inequality $2|ab| \le \delta a^2 + \delta^{-1}b^2$ for any $a,b\in\mathbb{R},\delta>0$ was used to pass from the second to the third line, and we invoked Lemma 6.7 in the final estimate.

Integrating by parts twice and again using Young's inequality, we find the estimate

$$\int_Q |\Delta u|^2\varphi^2\,dx \ge \frac{1}{2}\int_Q |\nabla^2 u|^2\varphi^2\,dx - C\int_Q |\nabla u|^2|\nabla\varphi|^2\,dx\ .$$

Hence, if we choose $\varepsilon_1 > 0$ such that $C_1\varepsilon_1 \le \frac{1}{8}$, we obtain

$$\int_Q |\nabla^2 u|^2\varphi^2 \, dx\, dt \le 4\int_\Sigma |\nabla u_0|^2\varphi^2 \, dx + CR^{-2}\int_0^T \int_{B_{2R}} |\nabla u|^2 \, dx\, dt$$
$$\le C\, E_0\big(1 + \frac{T}{R^2}\big) \, .$$

By choice of φ this implies the assertion of the lemma. □

As final preparation we show that if $u \in H^{1,2}$ weakly solves (6.1) on some interval $]0,T[$ with $|\nabla^2 u| \in L^2$, then u is smooth and smoothly extends up to time T.

6.11 Lemma. *Let $Q = B_R\times]0,T[$. If $u \in H^{1,2}(Q;N)$ solves (6.1) on Q with $|\nabla^2 u| \in L^2(Q)$ and with $\sup_{0<t<T} E\big(u(t);B_R\big) \le E_0$, then $u \in C^\infty(Q;N)$ and u extends smoothly to $B_R\times]0,T]$.*

Proof. In order to avoid using cut-off functions, we consider the case $\Sigma = T^2$, $Q = \Sigma\times]0,T[$, for simplicity.

In a first step we then show that $\nabla u \in L^q\big(\Sigma \times [t_*,T]\big)$ for any $q < \infty$ and any $t_* > 0$. We proceed by Moser iteration. For any $L \ge 1$ let $|\nabla u|_L = \min\{|\nabla u|, L\}$. Multiply (6.1) by $-\operatorname{div}\big(\nabla u|\nabla u|_L^s\big)$ and integrate by parts at any (fixed) time $t < T$. Dropping the term

$$\frac{s}{4}\int_\Sigma \nabla(|\nabla u|^2)\cdot\nabla(|\nabla u|_L^2)|\nabla u|_L^{s-2} \, dx = \frac{s}{4}\int_\Sigma |\nabla(|\nabla u|_L^2)|^2|\nabla u|_L^{s-2} \, dx \ge 0\,,$$

on account of (6.4) for any $s \ge 0$ we obtain the estimate

$$\int_\Sigma \Big(\partial_t\big(\frac{|\nabla u|^2}{2}\big)|\nabla u|_L^s + |\nabla^2 u|^2|\nabla u|_L^s\Big) \, dx \le C\int_\Sigma |\nabla^2 u||\nabla u|_L^s|\nabla u|^2 \, dx$$
$$\le \frac{1}{2}\int_\Sigma |\nabla^2 u|^2|\nabla u|_L^s \, dx + C\int_\Sigma |\nabla u|^4|\nabla u|_L^s \, dx\,.$$

The first term on the right is easily absorbed in the left hand side. Denoting $w_{L,s} = |\nabla u|_L^{s/2}|\nabla u|$, upon integrating in t, for any $0 < t_0 < t_1 < T$ we have

$$\int_{t_0}^{t_1}\int_\Sigma \partial_t\big(|\nabla u|^2\big)|\nabla u|_L^s \, dx\, dt = \int_\Sigma |w_{L,s}(t_1)|^2 \, dx - \int_\Sigma |w_{L,s}(t_0)|^2 \, dx$$
$$- \frac{s}{s+2}\int_{t_0}^{t_1}\int_\Sigma \partial_t\big(|\nabla u|_L^{s+2}\big) \, dx\, dt$$
$$\ge \frac{2}{s+2}\int_\Sigma |w_{L,s}(t_1)|^2 \, dx - \int_\Sigma |w_{L,s}(t_0)|^2 \, dx\,.$$

Thus, with constants depending only on Σ, N, and s we find

$$\begin{aligned}
&\sup_{t_0<t<t_1}\int_\Sigma |w_{L,s}|^2\,dx+\int_{t_0}^{t_1}\int_\Sigma |\nabla w_{L,s}|^2\,dx\,dt\\
&\quad\le C\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^2|w_{L,s}|^2\,dx\,dt+C\int_\Sigma |w_{L,s}(t_0)|^2\,dx\\
&\quad\le C\left(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^4\,dx\,dt\cdot\int_{t_0}^{t_1}\int_\Sigma |w_{L,s}|^4\,dx\,dt\right)^{\frac12}+C\int_\Sigma |w_{L,s}(t_0)|^2\,dx\,.
\end{aligned}$$

But by Lemma 6.7 for any $L\ge 1$, $s\ge 0$ on $Q_0=\Sigma\times[t_0,t_1]$ we can bound

$$\|w_{L,s}\|^4_{L^4(Q_0)}\le c_0\sup_{t_0\le t\le t_1}\|w_{L,s}(t)\|^2_{L^2(\Sigma)}\cdot\left(\|\nabla w_{L,s}\|^2_{L^2(Q_0)}+\|w_{L,s}\|^2_{L^2(Q_0)}\right).$$

Provided that $t_1\le t_0+1$, this yields the estimate

$$\int_{Q_0}|w_{L,s}|^4dx\,dt\le c_0\left(\sup_{t_0<t<t_1}\int_\Sigma |w_{L,s}|^2\,dx+\int_{Q_0}|\nabla w_{L,s}|^2dx\,dt\right)^2.$$

Together with our previous bounds we then obtain

$$\begin{aligned}
\int_{t_0}^{t_1}\int_\Sigma |w_{L,s}|^4dx\,dt\le c_1\int_{t_0}^{t_1}\int_\Sigma|\nabla u|^4\,dx\,dt\cdot\int_{t_0}^{t_1}\int_\Sigma|w_{L,s}|^4\,dx\,dt\\
+C\left(\int_\Sigma|w_{L,s}(t_0)|^2\,dx\right)^2.
\end{aligned}$$

Since u has finite energy and $|\nabla^2u|\in L^2(Q)$, Lemma 6.7 in particular yields that $|\nabla u|\in L^4(Q)$. By absolute continuity of the Lebesgue integral we may then choose $\gamma=\gamma(s,\Sigma,N)>0$ such that for $0\le t_0<t_1\le T$ with $t_1-t_0<\gamma$ there holds

$$2c_1\int_{t_0}^{t_1}\int_\Sigma|\nabla u|^4\,dx\,dt\le 1\,.$$

For any $L\ge 1$, any $s\ge 0$, and any $0<t_0<t_1<t_0+\gamma$, with a constant C depending only on the geometry and s there then results the estimate

$$\int_{t_0}^{t_1}\int_\Sigma|w_{L,s}|^4\,dx\,dt\le C\left(\int_\Sigma|w_{L,s}(t_0)|^2\,dx\right)^2.$$

Let $s_0=0$, $s_i=2(1+s_{i-1})$, $i\in\mathbb{N}$. Given $t_*>0$, suppose that $|\nabla u(t_{i-1})|\in L^{2+s_{i-1}}(\Sigma)$ for some $i\in\mathbb{N}$, where $0=t_0\le t_{i-1}<t_*$. We may assume that $t_*<\min\{t_{i-1}+\gamma,T\}$. Upon letting $L\to\infty$ then we can bound

$$\int_{t_{i-1}}^{t_*}\int_\Sigma|\nabla u|^{2+s_i}\,dx\,dt\le C\left(\int_\Sigma|\nabla u(t_{i-1})|^{2+s_{i-1}}\,dx\right)^2.$$

Covering the interval $]t_{i-1},T[$ with finitely many intervals of length γ, we find that $\nabla u\in L^{2+s_i}\big(\Sigma\times[t_*,T]\big)$. Moreover, by Fubini's theorem, there exists $t_i\in]t_{i-1},t_*[$ such that $|\nabla u(t_i)|\in L^{2+s_i}(\Sigma)$. Induction then yields that $|\nabla u(t_i)|\in$

$L^{2+s_i}(\Sigma)$ for suitable $0 \le t_i < t_*$ for any $i \in \mathbb{N}$, and we conclude that $\nabla u \in L^q\big(\Sigma \times [t_*, T]\big)$ for any $q < \infty$ and any $t_* > 0$, as claimed.

The linear theory for (6.4) then yields that $|u_t|, |\nabla^2 u| \in L^q\big(\Sigma \times [t_*, T]\big)$ for any $q < \infty$, and ∇u is uniformly Hölder continuous on $\Sigma \times [t_*, T]$, for any $t_* > 0$; see Ladyzhenskaya-Solonnikov-Ural'ceva [1; Theorem IV.9.1 and Lemma II.3.3]. Higher regularity up to $t = T$ then follows from the Schauder estimates for the heat equation, applied to (6.1), and the usual bootstrap argument. □

Proof of Theorem 6.6. For simplicity, we always consider the case $\Sigma = T^2$.
(1°) We first consider smooth initial data $u_0 \in C^\infty(\Sigma; N)$. By the local solvability of ordinary differential equations in Banach spaces, problem (6.1), (6.2) has a local solution $u \in C^\infty(\Sigma \times [0, T[; N)$; see Hamilton [1; p. 122 ff.]. By Lemma 6.8 we have $u_t \in L^2(\Sigma \times [0, T])$ and $E\big(u(t)\big) \le E\big(u(s)\big) \le E(u_0)$ for all $0 \le s \le t \le T$.

Moreover, if $R > 0$ can be chosen such that

$$\sup_{x \in \Sigma,\, 0 \le t \le T} E\big(u(t); B_R(x)\big) < \varepsilon_1 \ ,$$

it follows from Lemma 6.10 that

$$\int_0^T \int_\Sigma |\nabla^2 u|^2 \, dx \, dt \le c\, E(u_0) \left(1 + \frac{T}{R^2}\right) \ ,$$

and hence from Lemma 6.11 that u extends to a C^∞-solution of (6.1) on the closure $\Sigma \times [0, T]$. But then u can be extended beyond T as a smooth solution of (6.1). Thus, if $T > 0$ is maximal, there exist points $x_1, x_2, \ldots$ such that

$$\limsup_{t \nearrow T} \ E\big(u(t), B_R(x_k)\big) \ge \varepsilon_1$$

for any $R > 0$ and any index k. Choose any finite collection x_k, $1 \le k \le K$, of such points and for any $R > 0$, $k = 1, \ldots, K$, let $t_k < T$ be chosen such that

$$E\big(u(t_k); B_R(x_k)\big) \ge \varepsilon_1/2 \, .$$

We may assume that $t_k \ge T - \frac{\varepsilon_1 R^2}{4 c_1 E_0} =: t_0 \ge 0$ and that $B_{2R}(x_k) \cap B_{2R}(x_j) = \emptyset$ for $j \ne k$. With the constant c_1 from Lemma 6.9 then we have

$$\begin{aligned} E\big(u(t_0)\big) &\ge \sum_{k=1}^K E\big(u(t_0); B_{2R}(x_k)\big) \\ &\ge \sum_{k=1}^K \left(E\big(u(t_k); B_R(x_k)\big) - c_1 \frac{t_k - t_0}{R^2} E_0 \right) \\ &\ge K \left(\varepsilon_1/2 - c_1 \frac{T - t_0}{R^2} E_0 \right) = K \varepsilon_1 / 4 \, . \end{aligned} \tag{6.6}$$

Since by Lemma 6.8 we have $E\big(u(t_0)\big) \le E(u_0)$, this gives an upper bound

$$K \leq 4E(u_0)/\varepsilon_1$$

for the number K of singular points $x_1, \ldots, x_K$ at time T. Moreover for any $Q \subset\subset \Sigma \times [0,T] \setminus \{(x_1,T), \ldots, (x_K,T)\}$ there exists $R = R_Q > 0$ such that

$$\sup_{(x,t)\in Q} E\big(u(t); B_R(x)\big) < \varepsilon_1$$

and by Lemma 6.10, 6.11 our solution u extends to a C^∞-solution of (6.1) on $\Sigma \times [0,T] \setminus \{(x_1,T), \ldots (x_K,T)\}$.

(2°) For initial data $u_0 \in H^{1,2}(\Sigma; N)$ choose a sequence $u_{0m} \in C^\infty(\Sigma; N)$ approximating u_0 in $H^{1,2}(\Sigma; N)$. This is possible by Theorem 6.2. For each m let u_m be the associated solution of (6.1) with $u_m(0) = u_{0m}$ and $T_m > 0$ its maximal time of existence. Let $E_0 = sup_m E(u_{0m})$. Choose $R_0 > 0$ such that

$$\sup_{x\in\Sigma} E\big(u_0; B_{2R_0}(x)\big) \leq \varepsilon_1/4 \ .$$

This inequality will also hold with $\varepsilon_1/2$ instead of $\varepsilon_1/4$ for u_{0m} if $m \geq m_0$. By Lemma 6.9 then for $T = \frac{\varepsilon_1 R_0^2}{4c_1 E_0}$ we have

$$\begin{aligned}\sup_{\substack{x\in\Sigma \\ 0\leq t\leq \min\{T_m, T\}}} E\big(u_m(t); B_{R_0}(x)\big) &\leq \sup_{x\in\Sigma} E\big(u_{0m}, B_{2R_0}(x)\big) + c_1 \frac{T}{R_0^2} E_0 \\ &\leq \varepsilon_1/2 + \varepsilon_1/4 < \varepsilon_1 \ .\end{aligned}$$

By Lemma 6.10 it follows that $\nabla^2 u_m$ is uniformly bounded in $L^2\big(\Sigma \times [0,t]\big)$ for $t \leq \min\{T, T_m\}$ in terms of $E(u_0)$ and R_0 only. By Lemma 6.11, therefore, the interval of existence of u_m is both open and closed in $[0,T]$; that is, $T_m \geq T > 0$. Moreover

$$\int_0^T \int_\Sigma |\nabla^2 u_m|^2 \, dx \, dt \leq c\, E_0 \left(1 + \frac{T}{R_0^2}\right)$$

uniformly. Also using Lemma 6.8, we may assume that u_m converges weakly to a solution u of (6.1), (6.2) with $|u_t|, \ |\nabla^2 u|^2 \in L^2\big(\Sigma \times [0,T]\big)$, and such that

$$E\big(u(t)\big) \leq \liminf_{m\to\infty} E\big(u_m(t)\big) \leq \lim_{m\to\infty} E(u_{0m}) = E(u_0) \, ,$$

uniformly in $t \in [0,T]$. Since $|u_t| \in L^2\big(\Sigma \times [0,T]\big)$, the solution u attains its initial data u_0 continuously in $L^2(\Sigma; N)$; by the uniform energy bound $E\big(u(t)\big) \leq E(u_0)$, moreover, this is also true in the $H^{1,2}(\Sigma; N)$-topology.

By Lemma 6.11 we have $u \in C^\infty(\Sigma \times]0, T_1[, N)$ for some maximal $T_1 > T$. By Lemma 6.8, we then also have the energy estimate $E\big(u(t)\big) \leq E\big(u(s)\big) \leq E(u_0)$ for $0 \leq s \leq t < T_1$. By part (1°) of this proof, u extends smoothly to $\Sigma \times]0, T_1] \setminus \{(x_1, T_1), \ldots, (x_{K_1}, T_1)\}$ for some finite collection of singular points x_k, $1 \leq k \leq K_1$. Moreover, as $t \nearrow T_1$ we have $u(t) \to u_0^{(1)} \in H^{1,2}(\Sigma; N)$ weakly and strongly in $H^{1,2}_{loc}\big(\Sigma \setminus \{x_1, \ldots, x_{K_1}\}; N\big)$. Thus, by (6.6),

$$
\begin{aligned}
E(u_0^{(1)}) &= \lim_{R\to 0} E\big(u_0^{(1)}; \Sigma \setminus \bigcup_{k=1}^{K_1} B_{2R}(x_k)\big) \\
&= \lim_{R\to 0} \lim_{t\nearrow T_1} \left(E\big(u(t);\Sigma\big) - \sum_{k=1}^{K_1} E\big(u(t); B_{2R}(x_k)\big) \right) \\
&\le E(u_0) - K_1\varepsilon_1/4 \,.
\end{aligned}
$$

Letting $T_0 = 0$, $u^{(0)} = u$, $u_0^{(0)} = u_0$, by iteration we now obtain a sequence $u^{(m)}$ of smooth solutions to (6.1) on $\Sigma\times]T_m, T_{m+1}[$ with initial data $u_0^{(m)}$ at $t = T_m$ and such that $u^{(m)}(t) \to u_0^{(m+1)}$ weakly in $H^{1,2}(\Sigma; N)$ as $t \nearrow T_{m+1}$. Moreover, $u^{(m)}$ has finitely many singularities $x_1^{(m)}, \ldots, x_{K_{m+1}}^{(m)}$ at $t = T_{m+1}$, where

$$
E\big(u_0^{(m+1)}\big) \le E\big(u_0^{(m)}\big) - K_{m+1}\varepsilon_1/4 \le \ldots \le E(u_0) - \sum_{l=1}^{m+1} K_l\varepsilon_1/4 \,, \tag{6.7}
$$

and $u^{(m)}(t) \to u_0^{(m+1)}$ smoothly away from $x_1^{(m)}, \ldots, x_{K_{m+1}}^{(m)}$ as $t \nearrow T_{m+1}$, for any $m \in \mathbb{N}$. In particular, the total number of singularities of the flows $u^{(m)}$ is finite, and $T_m = \infty$ for some $m \in \mathbb{N}$. Piecing the $u^{(m)}$ together, for any initial $u_0 \in H^{1,2}(\Sigma; N)$ we thus obtain a weak solution u to (6.1), (6.2) on $\Sigma\times]0,\infty[$ satisfying the energy inequality $E\big(u(t)\big) \le E\big(u(s)\big) \le E(u_0)$ for $0 \le s \le t < \infty$, attaining the initial data continuously in $H^{1,2}\big(\Sigma; N\big)$, and smooth on $\Sigma\times]0,\infty[$ up to finitely many points.

(3°) Asymptotics: If for some $T > 0$, $R > 0$ we have

$$
\sup_{x\in\Sigma,\, t>T} E\big(u(t); B_R(x)\big) < \varepsilon_1 \,,
$$

then, by Lemma 6.10, for any $t > T$ there holds

$$
\int_t^{t+1} \int_\Sigma |\nabla^2 u|^2 \, dx \, dt \le c\, E(u_0)\big(1 + R^{-2}\big)
$$

with a uniform constant $c = c(\Sigma, N)$, while by Lemma 6.8 we have

$$
\int_t^{t+1} \int_\Sigma |u_t|^2 \, dx \, dt \to 0 \,.
$$

Hence we may choose a sequence $t_m \to \infty$ such that $u(t_m) \to u_\infty$ weakly in $H^{2,2}(\Sigma; N)$, while $u_t(t_m) \to 0$ in L^2. Moreover, by the Rellich-Kondrakov theorem $u(t_m) \to u_\infty$ also strongly in $W^{1,p}(\Sigma; N)$ for any $p < \infty$. Testing (6.1) with $\Delta(u - u_\infty)$ and integrating by parts on $\Sigma \times \{t_m\}$, we obtain that

$$
\begin{aligned}
&\int_{\Sigma\times\{t_m\}} |\nabla^2(u - u_\infty)|^2 \, dx \\
&\le \left| \int_{\Sigma\times\{t_m\}} u_t \Delta(u - u_\infty) dx \right| + \left| \int_{\Sigma\times\{t_m\}} A(u)(\nabla u, \nabla u)\Delta(u - u_\infty) dx \right| \to 0 \,;
\end{aligned}
$$

that is, $u(t_m) \to u_\infty$ also strongly in $H^{2,2}(\Sigma; N)$. Passing to the limit $m \to \infty$ in (6.1), evaluated at t_m, we then find that u_∞ is harmonic.

It remains to study the case that for a sequence $s_m \to \infty$ there exist points $x_{0m} \in \Sigma$ such that for any $R > 0$ we have

$$\liminf_{m\to\infty} \big(E\big(u(s_m); B_R(x_{0m})\big)\big) > \varepsilon_1/2\,.$$

Passing to a subsequence, we may assume that $x_{0m} \to x_0$, and there holds

$$\liminf_{m\to\infty} \big(E\big(u(s_m); B_{2R}(x_0)\big)\big) \geq \varepsilon_1/2 \tag{6.8}$$

for any $R > 0$. Suppose that (6.8) holds for points $x_1, \ldots, x_K$. Choose $R > 0$ such that $B_{2R}(x_j) \cap B_{2R}(x_k) = \emptyset\,(j \neq k)$. Then for sufficiently large m we have

$$E\big(u(s_m)\big) \geq \sum_{k=1}^{K} E\big(u(s_m); B_{2R}(x_k)\big) \geq K\varepsilon_1/4\,. \tag{6.9}$$

It follows that $K \leq 4E(u_0)/\varepsilon_1$. We thus may assume that the collection $x_1, \ldots, x_K$ is maximal with property (6.8). By repeated selection of subsequences of (s_m) and in view of the uniform boundedness of the number K of concentration points (independent of the sequence (s_m)), we can even achieve that $x_1, \ldots, x_K$ is maximal with the property that

$$\limsup_{m\to\infty} \big(E\big(u(s_m); B_{2R}(x_k)\big)\big) \geq \varepsilon_1/2$$

for any $R > 0$, $1 \leq k \leq K$. By Lemma 6.9 then for any $\Omega \subset\subset \Sigma \setminus \{x_1, \ldots, x_K\}$ there exists $R > 0$ such that for any $x \in \Omega$ with $\tau = \frac{\varepsilon_1 R^2}{4c_1 E(u_0)}$ there holds

$$\sup_{s_m \leq t \leq s_m+\tau} E\big(u(t); B_R(x)\big) \leq \limsup_{m\to\infty} E\big(u(t_m); B_{2R}(x)\big) + \varepsilon_1/2 \leq \varepsilon_1$$

for large m. By compactness, $\overline{\Omega}$ is covered by finitely many such balls $B_R(x)$. Hence by Lemma 6.10 there holds

$$\int_{s_m}^{s_m+\tau} \int_\Omega |\nabla^2 u|^2 \, dx\, dt \leq c\, E(u_0) \left(1 + \frac{\tau}{R^2}\right) = c\, E(u_0) + c \tag{6.10}$$

for large m, while Lemma 6.8 implies that

$$\int_{s_m}^{s_m+\tau} \int_\Omega |u_t|^2 \, dx\, dt \to 0\,. \tag{6.11}$$

Exhausting $\Sigma \setminus \{x_1, \ldots, x_K\}$ by such domains Ω, we may choose a sequence $t_m \in [s_m, s_m + \tau]$ such that $u(t_m) \to u_\infty$ weakly in $H^{2,2}_{loc}(\Sigma \setminus \{x_1, \ldots, x_K\}; N)$, where $u_\infty \colon \Sigma \setminus \{x_1, \ldots, x_K\} \to N$ is harmonic. Moreover, by Lemma 6.8 we also may assume that $u(t_m) \to u_\infty$ weakly in $H^{1,2}(\Sigma; N)$, and u_∞ has finite energy. By the regularity result of Sacks-Uhlenbeck [1; Theorem 3.6], then u_∞ extends to a smooth harmonic map $u_\infty \in C^\infty(\Sigma; N)$.

(4°) Singularities: Suppose $(\bar{x}, \bar{t})$ is singular in the sense that for any $R > 0$ we have

$$\limsup_{t \nearrow \bar{t}} \; E\big(u(t); B_R(\bar{x})\big) \geq \varepsilon_1 \,,$$

or that $\bar{x}$ is a concentration point at $\bar{t} = \infty$ in the sense that for a sequence $s_m \to \infty$ as in (3°) for any $R > 0$ we have

$$\liminf_{m \to \infty} \big(E\big(u(s_m); B_{2R}(\bar{x})\big)\big) \geq \varepsilon_1/2 \,.$$

By finiteness of the singular set, $\bar{x}$ is isolated among concentration points. Hence there is $R_0 > 0$ such that when we let $R_m \to 0$ and set $\tau_m = \frac{\varepsilon_1 R_m^2}{16 c_1 E(u_0)}$, there are $\bar{t}_m \nearrow \bar{t}$ and $\bar{x}_m \to \bar{x}$ such that

$$E\big(u(\bar{t}_m); B_{R_m}(\bar{x}_m)\big) = \sup_{\substack{x \in B_{2R_0}(\bar{x}) \\ \bar{t}_m - \tau_m \leq t \leq \bar{t}_m}} E\big(u(t); B_{R_m}(x)\big) = \varepsilon_1/4 \,.$$

We may assume $B_{R_m}(\bar{x}_m) \subset B_{R_0}(\bar{x})$; moreover, if $\bar{t} = \infty$ we may suppose that $s_m \leq \bar{t}_m - \tau_m \leq s_m + \tau_m$. Rescale

$$u_m(x,t) := u(\bar{x}_m + R_m x, \bar{t}_m + R_m^2 t)$$

and note that $u_m \colon B_{1+R_0/R_m} \times [t_0, 0] \to N$, with $t_0 = -\frac{\varepsilon_1}{16 c_1 E(u_0)}$, solves (6.1) classically with

$$\sup_{\substack{R_m |x| \leq R_0 \\ t_0 \leq t \leq 0}} E\big(u_m(t); B_1(x)\big) \leq E\big(u_m(0); B_1\big) = \varepsilon_1/4$$

and

$$\int_{t_0}^{0} \int_{B_{R_0/R_m}} |u_{m,t}|^2 \, dx \, dt \leq \int_{\bar{t}_m - \tau_m}^{\bar{t}_m} \int_{\Sigma} |u_t|^2 \, dx \, dt \to 0 \,,$$

as $m \to \infty$. From Lemma 6.10 it follows that also

$$\int_{t_0}^{0} \int_{B_{R_0/R_m}} |\nabla^2 u_m|^2 \, dx \, dt \leq c$$

uniformly. Hence for a suitable sequence $t_m \in [t_0, 0]$ we have convergence $u_{m,t}(t_m) \to 0$ in L^2, while $u_m(t_m) \to \tilde{u}$ weakly in $H^{2,2}_{loc}(\mathbb{R}^2; N)$ and strongly in $H^{1,2}_{loc}(\mathbb{R}^2; N)$. (In fact, we can even show that $u_m(t_m) \to \tilde{u}$ strongly in $H^{2,2}_{loc}(\mathbb{R}^2; N)$.) Upon passing to the limit $m \to \infty$ in Equation (6.1) for u_m at $t = t_m$ we see that $\tilde{u}$ is harmonic. Moreover, by Lemma 6.9 with error $o(1) \to 0$ as $m \to \infty$ we have

$$\begin{aligned} E(\tilde{u}; B_2) &= E\big(u_m(t_m); B_2\big) - o(1) \\ &\geq E\big(u_m(0); B_1\big) + c_1 t_0 E(u_0) - o(1) = \varepsilon_1/4 - \varepsilon_1/16 - o(1) > 0 \end{aligned}$$

for sufficiently large m. Thus $\tilde{u} \not\equiv const.$ Finally,

$$E(\tilde{u}) \le \liminf_{m\to\infty} E(u_m(t_m); B_{R_0/R_m}) \le \liminf_{m\to\infty} E(u(\bar{t}_m + R_m^2 t_m)) \le E(u_0)\,,$$

and by the result of Sacks-Uhlenbeck [1; Theorem 3.6] quoted earlier, $\tilde{u}$ extends to a harmonic map $\bar{u}\colon S^2 \to N$.

In particular, we conclude $E(\bar{u}) = E(\tilde{u}) \ge \varepsilon_0$. Therefore, and using Lemma 6.9, for any $r, R > 0$ we obtain that

$$\begin{aligned} E\big(u(\bar{s}_m); B_{2R}(\bar{x}_m)\big) &\ge E\big(u(\bar{t}_m + R_m^2 t_m); B_R(\bar{x}_m)\big) - o(1) \\ &= E\big(u_m(t_m); B_{R/R_m}\big) - o(1) \ge E\big(u_m(t_m); B_r\big) - o(1) \\ &= E\big(\tilde{u}_m; B_r\big) - o(1) \ge \varepsilon_0 - o(1)\,, \end{aligned}$$

where $o(1) \to 0$ as we first let $m \to \infty$ and then $r \to \infty$. It follows that estimates (6.6), (6.7), and (6.9) may be improved, yielding the upper bound $K \le E(u_0)/\varepsilon_0$ for the total number of singularities and concentration points.

(5°) Uniqueness: It suffices to show that two solutions u, v of (6.1) satisfying

$$\partial_t u\ , \nabla^2 u\ , \partial_t v\ , \nabla^2 v \in L^2\big(\Sigma \times [0,T]\big)$$

and with $u\,|_{t=0} = u_0 = v\,|_{t=0}$ coincide. Let $w = u - v$. By (6.1), w satisfies the differential inequality

$$\begin{aligned} |\partial_t w - \Delta_\Sigma w| &= |A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v)| \\ &\le C|w|\big(|\nabla u|^2 + |\nabla v|^2\big) + C|\nabla w|\big(|\nabla u| + |\nabla v|\big)\,. \end{aligned}$$

Multiplying by w and integrating by parts over $\Sigma \times [0, t_0]$, for any $t_0 > 0$ we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Sigma |w(t_0)|^2\,dx + \int_0^{t_0}\int_\Sigma |\nabla w|^2\,dx\,dt \\ &\qquad \le C\int_0^{t_0}\int_\Sigma |w|^2\big(|\nabla u|^2 + |\nabla v|^2\big)\,dx\,dt + \frac{1}{2}\int_0^{t_0}\int_\Sigma |\nabla w|^2\,dx\,dt\,. \end{aligned} \tag{6.12}$$

Here we used Young's inequality to estimate

$$C|w||\nabla w|\big(|\nabla u| + |\nabla v|\big) \le \frac{1}{2}|\nabla w|^2 + C|w|^2\big(|\nabla u|^2 + |\nabla v|^2\big)\,,$$

and we also used the fact that $w\,|_{t=0} = 0$.

By Lemma 6.7, the functions $|\nabla u|$, $|\nabla v|$, and w belong to $L^4\big(\Sigma \times [0,t_0]\big)$, and for any $\delta > 0$ we can estimate

$$\begin{aligned} &\int_0^{t_0}\int_\Sigma \big(|\nabla u|^4 + |\nabla v|^4\big)\,dx\,dt \\ &\qquad \le CE(u_0)\int_0^{t_0}\int_\Sigma \big(|\nabla^2 u|^2 + |\nabla^2 v|^2 + |\nabla u|^2 + |\nabla v|^2\big)\,dx\,dt \le \delta^2\,, \end{aligned}$$

if $0 < t_0 \le T_0 = T_0(\delta)$ is sufficiently small. We may assume that $T_0 \le 1$. Then similarly for such t_0 we can bound

$$\int_0^{t_0}\int_\Sigma |w|^4\,dx\,dt$$
$$\le C\left(\sup_{0\le t\le t_0}\int_\Sigma |w(t)|^2\,dx\right)\cdot\left(\int_0^{t_0}\int_\Sigma \left(|\nabla w|^2+|w|^2\right)dx\,dt\right)$$
$$\le C\left(\sup_{0\le t\le t_0}\int_\Sigma |w(t)|^2\,dx+\int_0^{t_0}\int_\Sigma |\nabla w|^2\,dx\,dt\right)^2 .$$

Given $\delta>0$, choose $0<t_0\le T_0=T_0(\delta)$ such that

$$\int_\Sigma |w(t_0)|^2\,dx=\sup_{0\le t\le T_0}\int_\Sigma |w(t)|^2\,dx=\sup_{0\le t\le t_0}\int_\Sigma |w(t)|^2\,dx.$$

Then, from (6.12) we obtain the estimate

$$\sup_{0\le t\le t_0}\int_\Sigma |w(t)|^2\,dx+\int_0^{t_0}\int_\Sigma |\nabla w|^2\,dx\,dt$$
$$\le C\left(\int_0^{t_0}\int_\Sigma |w|^4\,dx\,dt\right)^{1/2}\left(\int_0^{t_0}\int_\Sigma \left(|\nabla u|^4+|\nabla v|^4\right)dx\,dt\right)^{1/2}$$
$$\le C_1\delta\left(\sup_{0\le t\le t_0}\int_\Sigma |w(t)|^2\,dx+\int_0^{t_0}\int_\Sigma |\nabla w|^2\,dx\,dt\right) .$$

with a constant $C_1=C_1(\Sigma,N)$. Thus, if we choose $\delta>0$ such that $C_1\delta<1$, it follows that $w\equiv 0$ on $\Sigma\times[0,T_0(\delta)]$. More generally, the above argument shows that the maximal interval $I\subset[0,T]$ containing $t=0$ and such that $u(t)=v(t)$ for $t\in I$ is relatively open. Since I is trivially closed, uniqueness follows. □

6.12 Remark. Chang [6] has obtained the analogue of Theorem 6.6 for the evolution problem (6.1), (6.2) on manifolds Σ with boundary $\partial\Sigma\neq\emptyset$ and with Dirichlet boundary data. His result – in the same way as we used Theorem 6.6 to prove Theorem 6.5 – can be used to prove, for instance, the existence and multiplicity results of Brezis-Coron [1] and Jost [1] for harmonic maps with boundary.

Finite-time Blow-up.

The question whether singularities actually may occur in finite time remained open for quite a while. It was finally settled by Chang-Ding-Ye [1] who gave the following example.

6.13 The co-rotational setting. Let $x=re^{i\varphi}$ be the representation of points $x\in\mathbb{R}^2$ in polar coordinates (r,φ). Similarly, we denote as θ the longitudinal angle on S^2 and let ρ denote the distance traveled along any fixed geodesic from the north pole N. A map $u\colon B=B_1(0)\to S^2$ is co-rotational (of degree 1), if u is represented by maps

$$\theta(re^{i\varphi}) = \varphi\,,\ \rho(re^{i\varphi}) = h(r)$$

so that

$$u(re^{i\varphi}) = \exp_N(h(r)e^{i\varphi})\,.$$

The energy of u then may be expressed as

$$E(u) = \frac{1}{2}\int_B |\nabla u|^2\,dx = \pi\int_0^1 \big(|h_r|^2 + \frac{sin^2 h}{r^2}\big)\,dr\,, \tag{6.13}$$

and u is harmonic if and only if h satisfies the equation

$$-\frac{1}{r}(rh_r)_r + \frac{\sin 2h}{2r^2} = 0 \text{ on } [0,1[\,.$$

Likewise, a smooth family $u(t)$, $0 \le t < T$, of such maps, represented by a map $h\colon [0,1]\times[0,T[\to \mathbb{R}$, evolves by the heat flow for harmonic maps from B to S^2, that is, solves the equation

$$u_t - \Delta u = u|\nabla u|^2 \text{ on } B\times[0,T[\,, \tag{6.14}$$

if and only if h solves the scalar evolution equation

$$h_t - \frac{1}{r}(rh_r)_r + \frac{\sin 2h}{2r^2} = 0 \text{ on } [0,1[\times[0,T[\,. \tag{6.15}$$

The representation

$$\bar h(r) = 2\arctan(r)$$

of the harmonic map $\bar u\colon \mathbb{R}^2 \to S^2$ obtained as the inverse of stereographic projection then is a stationary (time-independent) solution of (6.15), as are all functions

$$\bar h_\lambda(r) = \bar h(r/\lambda)\,,\ \lambda > 0\,,$$

corresponding to the scaled maps $\bar u_\lambda(x) = \bar u(x/\lambda)$, $\lambda > 0$.

Chang-Ding-Ye [1] show the following result.

6.14 Theorem. *For any smooth co-rotational data $u_0\colon B \to S^2$, represented by a map $h_0\colon [0,1] \to \mathbb{R}$ with $h_0(0) = 0$ and $h_0(1) = b > \pi$, the solution u to the Equation (6.14) with inital and boundary data u_0, or, equivalently, the solution h to Equation (6.15) with data*

$$h(0,r) = h_0(r)\,,\ h(t,0) = h_0(0) = 0\,,\ h(t,1) = h_0(1) = b\,,$$

will develop a singularity at a point $(0,T)$ for some $T < \infty$ in the sense that $|\nabla u|$ and $|h_r|$ will be unbounded in any neighborhood of the point $(0,T) \in B\times[0,T]$.

Observe that all maps h_0 appearing in the Chang-Ding-Ye result cover the sphere slightly more than once. It is an interesting open question what the precise rate of blow-up is.

Proof. Let $b > \pi$. We show that (6.15) admits a smooth sub-solution f with

$$f(0,t) = 0 \le f(r,t) \le f(1,t) \le b$$

on $[0,1] \times [0, T_f[$ for some $T_f < \infty$ such that

$$f_r(0,t) \to \infty \text{ as } t \to T_f\,.$$

For any data h_0 as in the theorem such that $h_0 \ge f(\cdot, 0)$ we then consider the corresponding solution h of (6.15) with initial data $h(0) = h_0$. Provided that a smooth solution h exists for $0 \le t < T_f$, by the maximum principle h will satisfy $h \ge f$ on $[0,1] \times [0, T_f[$. Since $h(0,t) = 0 = f(0,t)$, it then also follows that

$$h_r(0,t) \to \infty \text{ as } t \to T_f\,.$$

Hence h must blow up at some time $T \le T_f$.

For constants $\varepsilon > 0$, $\mu > 0$ consider the function $l = l(r)$, given by

$$l(r) = \bar{h}_\mu(r^{1+\varepsilon})\,.$$

Note that l satisfies the equation

$$-\frac{1}{r}(rl_r)_r + \frac{(1+\varepsilon)^2 \sin 2l}{2r^2} = 0\,. \tag{6.16}$$

Fix some number $0 < \varepsilon < 1$ once and for all. Given $b > \pi$ as above, we can fix $\mu > 0$ sufficiently large so that we have $l(1) + \pi \le b$ and so that, in addition, there holds

$$\cos l(r) \ge \frac{1}{1+\varepsilon} \text{ for } 0 \le r \le 1\,. \tag{6.17}$$

With a function $\lambda = \lambda(t) > 0$ to be specified later we then make the ansatz $f = \bar{h}_\lambda + l$. We have $0 \le f(r,t) \le b$ for $0 \le r \le 1$ and all $t \ge 0$ where $\lambda(t)$ is defined. Moreover, in view of (6.15), (6.16) we compute

$$\tau(f) := \frac{1}{r}(rf_r)_r - \frac{\sin 2f}{2r^2} = \frac{\sin 2\bar{h}_{\lambda(t)} - \sin 2f + (1+\varepsilon)^2 \sin 2l}{2r^2}\,. \tag{6.18}$$

Now observe that for any α, $\beta \in \mathbb{R}$ there holds

$$\begin{aligned}\sin 2(\alpha+\beta) - \sin 2\alpha &= \sin((2\alpha+\beta)+\beta) - \sin((2\alpha+\beta)-\beta)\\ &= 2\cos(2\alpha+\beta)\sin\beta\,.\end{aligned}$$

Thus we may simplify (6.18) and use (6.17) to obtain

$$\begin{aligned}\tau(f) &= r^{-2}\big((1+\varepsilon)^2\cos l - \cos(2\bar{h}_{\lambda(t)} + l)\big)\sin l\\ &\ge r^{-2}\big(1+\varepsilon - \cos(2\bar{h}_{\lambda(t)} + l)\big)\sin l \ge r^{-2}\varepsilon\,\sin l\,.\end{aligned}$$

But for $|\alpha| < \pi/2$ we have the identity

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}\,.$$

Since we may assume that $\mu > 1$, it follows that

$$\tau(f) \geq \frac{\varepsilon}{r^2}\,\sin l = \frac{\varepsilon}{r^2}\,\sin(2\arctan(r^{1+\varepsilon}/\mu)) = \frac{\varepsilon}{r^2}\,\frac{2\mu r^{1+\varepsilon}}{\mu^2 + r^{2(1+\varepsilon)}} \geq \varepsilon_1 r^{\varepsilon-1}\,,$$

where $\varepsilon_1 = \frac{2\mu\varepsilon}{\mu^2+1} > 0$. Given $\lambda_0 > 0$, $\delta > 0$ define

$$\lambda(t) = \left(\lambda_0^{1-\varepsilon} - (1-\varepsilon)\delta t\right)^{1/(1-\varepsilon)}$$

so that

$$\frac{d}{dt}\lambda = -\delta\lambda^{\varepsilon}\,.$$

Then we compute

$$\frac{d}{dt}f = \frac{d}{dt}\bar{h}_{\lambda(t)} = -\frac{2r}{r^2+\lambda^2}\,\frac{d}{dt}\lambda = \frac{2\delta r\lambda^{\varepsilon}}{r^2+\lambda^2}\,.$$

Hence we obtain

$$\frac{d}{dt}f - \tau(f) \leq \frac{2\delta\lambda^{\varepsilon} r}{r^2+\lambda^2} - \varepsilon_1 r^{\varepsilon-1} = \left(\frac{2\delta\lambda^{\varepsilon} r^{2-\varepsilon}}{r^2+\lambda^2} - \varepsilon_1\right) r^{\varepsilon-1}\,.$$

But by Young's inequality we have

$$\lambda^{\varepsilon} r^{2-\varepsilon} \leq C(\varepsilon)(r^2+\lambda^2)\,.$$

Hence, for any $\varepsilon > 0$ there is $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ the function f is a sub-solution to (6.15), as desired.

Observe that the constants μ and λ_0 in the above construction may be chosen arbitrarily large. Thus, for any h_0 with $h_0(1) > \pi$ and such that $\frac{dh_0}{dr}(0) > 0$ while $h_0(r) > 0$ for $r > 0$ we can arrange that $f(\cdot, 0) \leq h_0$ and conclude that the solution h to (6.15) must blow up in finite time.

By the maximum principle the above conditions will hold true at any time $t > 0$ for the solution $h(t)$ issuing from any function $h_0 \geq 0$ which satisfies the hypotheses of the theorem.

Finally, observe that for any $k \in \mathbb{N}$ the function $f - k\pi$ again is a sub-solution to (6.15). By iteratively employing the sub-solutions $f - k\pi$ for $k = k_0, \ldots, 1$, where k_0 is determined so that $k_0\pi + h_0 \geq 0$, for a general function h_0 as in the theorem we then see that after some waiting-time $t_0 > 0$ the condition $h(t) \geq 0$ will hold true for all $t \geq t_0$, provided the solution h does not blow up before t_0. The proof is complete. □

Reverse Bubbling and Nonuniqueness

The fact that the energy $E(u(t))$ of the solutions u constructed in Theorem 6.6 to the heat flow (6.1), (6.2) is non-increasing in t is essential for the uniqueness of these solutions. In fact, and quite surprisingly, prompted by the work of Chang-Ding-Ye [1] presented above, Topping [1] and Bertsch et al. [1] independently constructed examples of weak solutions to (6.1), (6.2) which violate the energy inequality and therefore differ from the solutions constructed in Theorem 6.6. These solutions, moreover, are smooth with isolated singularities and have bounded energy.

The construction intuitively is achieved by attaching more and more concentrated "bubbles" to some smooth given data u_0 at time $t = 1$, and letting the new configuration evolve by the flow, say, for time $1 < t < 2$. Under suitable hypotheses on u_0, the resulting sequence of solutions to (6.1) then can be shown to converge to a limit u which blows up as we go backwards to the initial time $t = 1$ where a "bubble" develops. This process can therefore best be termed "reverse bubbling".

The following result is taken from Topping [1], Theorem 1.1. We again work in the co-rotational setting described in Section 6.12. Recall the representation

$$\bar{h}(r) = 2\arctan(r)$$

of the harmonic map obtained as the inverse of stereographic projection and the notation

$$\bar{h}_\lambda(r) = \bar{h}(r/\lambda), \lambda > 0$$

for the rescaled map $\bar{h}$.

6.15 Theorem. *Let $u_0 \colon B \to S^2$ be the co-rotational harmonic map represented by the function $h_0(r) = \pi - \overline{h}(r)$, thus precisely covering the lower hemisphere. There exists a co-rotational weak solution $u \in H^1_{loc}(B \times [0,\infty[; S^2)$ to (6.1) such that $u(t) = u_0$ for $t \le 1$ and $u(t) \neq u_0$ for $t > 1$. Moreover, u is smooth except at the point $(0,1) \in B \times [0,\infty[$, and there holds the uniform energy bound*

$$2\pi = E(u_0) \le E(u(t)) \le E(u_0) + 4\pi = \lim_{t \downarrow 1} E(u(t)) \quad \text{for all } t\,. \tag{6.19}$$

Proof. Similar to the proof of Theorem 6.14 we first construct a super-solution g to the flow (6.15) which exhibits reverse bubbling at the time $t = 0$. Many details of the construction may be borrowed from the preceding section.

For constants $\varepsilon > 0$, $\mu > 0$ again consider the function $l = l(r)$, given by

$$l(r) = \bar{h}_\mu(r^{1+\varepsilon})\,.$$

Recall that l satisfies Equation (6.16), that is,

$$-\frac{1}{r}(rl_r)_r + \frac{(1+\varepsilon)^2 \sin 2l}{2r^2} = 0\,.$$

We fix $0 < \varepsilon < 1$ and then choose $\mu > 0$ sufficiently large so that the estimates $l(1) < \pi/4$ and (6.17) both hold, that is,

$$\cos l(r) \geq \frac{1}{1+\varepsilon} \text{ for } 0 \leq r \leq 1\,.$$

For suitable $\delta > 0$, to be determined in the sequel, we then define

$$\lambda(t) = \big((1-\varepsilon)\delta t\big)^{1/(1-\varepsilon)}\,,$$

satisfying

$$\frac{d}{dt}\lambda = \delta\lambda^{\varepsilon},\ \lambda(0) = 0.$$

We then make the ansatz

$$g = \bar{h}_\lambda - l\,.$$

In the identity

$$\sin 2(\alpha+\beta) - \sin 2\alpha = 2\cos(2\alpha+\beta)\sin\beta\,,$$

we now let $\alpha = \bar{h}_{\lambda(t)}$, $\beta = -l$. By means of (6.16), (6.17) we then compute

$$\begin{aligned}
\tau(g) = \frac{1}{r}(rg_r)_r - \frac{\sin 2g}{2r^2} &= \frac{\sin 2\bar{h}_{\lambda(t)} - \sin 2g - (1+\varepsilon)^2 \sin 2l}{2r^2} \\
&= r^{-2}\big(\cos(2\bar{h}_{\lambda(t)} - l) - (1+\varepsilon)^2\cos l\big)\,\sin l \\
&\leq r^{-2}\big(\cos(2\bar{h}_{\lambda(t)} - l) - (1+\varepsilon)\big)\,\sin l \\
&\leq -\frac{\varepsilon}{r^2}\,\sin l = -\frac{2\varepsilon\mu r^{\varepsilon-1}}{\mu^2 + r^{2(1+\varepsilon)}} \leq -\varepsilon_1 r^{\varepsilon-1}\,,
\end{aligned}$$

where $\varepsilon_1 = \frac{2\mu\varepsilon}{\mu^2+1} > 0$. Moreover, we have

$$\frac{d}{dt}g = \frac{d}{dt}\bar{h}_{\lambda(t)} = -\frac{2r}{r^2+\lambda^2}\frac{d}{dt}\lambda = -\frac{2\delta r\lambda^{\varepsilon}}{r^2+\lambda^2}\,.$$

Hence we obtain

$$\frac{d}{dt}g - \tau(g) \geq \varepsilon_1 r^{\varepsilon-1} - \frac{2\delta\lambda^{\varepsilon} r}{r^2+\lambda^2} = \left(\varepsilon_1 - \frac{2\delta\lambda^{\varepsilon} r^{2-\varepsilon}}{r^2+\lambda^2}\right) r^{\varepsilon-1} \geq 0\,,$$

provided we choose $0 < \delta < \delta_0$ sufficiently small.

Next, for $n \in \mathbb{N}$ consider the co-rotational solutions u_n to the flow (6.14) with initial data u_{0n} represented by the functions

$$h_{0n}(r) = \min\{h_0(r), \bar{h}(nr)\}\,,\ 0 \leq r \leq 1\,.$$

Note that there is a unique point $0 < r_n < 1$, tending monotonically to 0 as $n \to \infty$, such that $h_{0n}(r) = \bar{h}(nr)$ for $r < r_n$ and $h_{0n}(r) = h_0(r)$ for $r > r_n$; in particular, we have that $h_{0n}(0) = \bar{h}(0) = 0$ and $h_{0n}(1) = h_0(1) = \pi/2$ for all

n. Moreover, there holds $h_{0m}(r) \le h_{0n}(r)$ for all $r \in [0,1]$, whenever $m \le n$. Finally, from the representation (6.13) we also conclude the bound

$$E(u_{0n}) = E(u_0, B \setminus B_{r_n}) + E(\bar{u}, B_{nr_n}) < E(u_0) + E(\bar{u}) = 6\pi = \lim_{n\to\infty} E(u_{0n})\,.$$

Here, in addition, we use the fact that u_0 is conformal; its Dirichlet energy therefore equals the area 2π covered by its image.

By the energy inequality, Lemma 6.8, we then have

$$E(u_n(t)) < E(u_0) + E(\bar{u}) = 6\pi \tag{6.20}$$

for all $t > 0$. In particular, the flow solutions u_n will be smooth for $t > 0$, as any singularity, say at a time $t_n > 0$, by Theorem 6.6 would absorb at least the energy 4π and therefore not leave enough energy for the weak limit $u_n(t_n)$ to span the boundary data given by u_0, which would require at least energy 2π.

Letting h_n be the solution to (6.15) representing u_n, we thus may invoke the maximum principle to conclude that

$$0 \le h_m(r,t) \le h_n(r,t) \le \min\{h_0(r), g(r,t)\} \tag{6.21}$$

for all $r \in [0,1]$, $t \ge 0$, whenever $m \le n$. Indeed, since h_0 is a time-independent solution of (6.15) and since we have $h_{0m} \le h_{0n} \le h_0$ for $m \le n$, the first inequalities are immediate. Moreover, from our assumption that $l(1) < \pi/4$ we deduce that $h_0(r) < \pi - l(r) = \lim_{t\downarrow 0} g(r,t)$ for any $r > 0$. Thus for any $n \in \mathbb{N}$ there exists a number $\tau > 0$ such that $h_{0n} \le g(\tau)$ and hence $h_n(t) \le g(\tau + t)$ for all $t \ge 0$. But by construction the function $g(t)$ is non-increasing in t, and (6.21) follows.

In view of the unifom energy bound (6.20) and the pointwise bounds (6.21) a subsequence h_n thus converges smoothly away from $r = 0$ to a solution h of (6.15). We can strengthen this assertion to include the axis $r = 0$ by means of Lemma 6.10 once we show the following result.

6.16 Lemma. *There is a time $T > 0$ and an index $N \in \mathbb{N}$ such that for any $t_0 \in]0, T[$ there exists a radius $r_0 > 0$ so that*

$$\sup_{t_0 \le t \le T} E(u_n(t), B_{r_0}(0)) < \varepsilon_1 \quad \textit{for all } n \ge N\,,$$

where $\varepsilon_1 > 0$ is the constant determined in Lemma 6.10 for $N = S^2$.

Proof. We may of course assume that $\sqrt{\frac{\varepsilon_1}{4\pi}} < \frac{\pi}{2}$.

Since $\sup_{0\le r\le 1} h_{0n}(r) \to \pi$ as $n \to \infty$, we can find an index N and a radius $r_1 := r_N > 0$ such that

$$h_{0N}(r_1) = \sup_{0\le r\le 1} h_{0N}(r) \ge \pi - \frac{1}{2}\sqrt{\frac{\varepsilon_1}{4\pi}}\,.$$

We may then fix $T > 0$ sufficiently small to insure the inequality

$$h_N(r_1,t) \geq \pi - \sqrt{\frac{\varepsilon_1}{4\pi}}$$

for any $t \in [0,T]$. By (6.21) we then also have

$$h_n(r_1,t) \geq \pi - \sqrt{\frac{\varepsilon_1}{4\pi}} \tag{6.22}$$

for any $n \geq N$, uniformly in $t \in [0,T]$. Reducing T, if necessary, we also may assume that there holds $g(1,t) > \pi/2$ for $0 \leq t \leq T$.

For any $t_0 \in]0,T[$ now we choose $0 < r_0 < 1$ so that

$$g(r,t_0) < \sqrt{\frac{\varepsilon_1}{4\pi}}$$

for $r \in [0,r_0]$. Recalling that $g(r,t)$ is monotonically decreasing in t for any r, by (6.21) then for any $r \in [0,r_0]$, any $t \in [t_0,T]$, and any $n \geq N$ we have

$$h_n(r,t) \leq g(r,t) \leq g(r,t_0) < \sqrt{\frac{\varepsilon_1}{4\pi}}\,. \tag{6.23}$$

In particular, in view of (6.22) and our assumption about ε_1 then it follows that $0 < r_0 < r_1 < 1$. Since for each $t \in [t_0,T]$ the map $h_n(t)$, restricted to $[r_0,r_1]$ thus covers the interval $[\sqrt{\frac{\varepsilon_1}{4\pi}}, \pi - \sqrt{\frac{\varepsilon_1}{4\pi}}]$, we conclude that the map $u_n(t)$, restricted to the annulus $B_{r_1} \setminus B_{r_0}$ covers the sphere with caps of radius $\sqrt{\frac{\varepsilon_1}{4\pi}}$ removed from each pole. In particular, since the area of a geodesic disk on S^2 of radius $r > 0$ is less than πr^2, the area covered by $u_n(t)$ over this region is larger than $4\pi - 2\pi\big(\sqrt{\frac{\varepsilon_1}{4\pi}}\big)^2 = 4\pi - \frac{\varepsilon_1}{2}$, and hence

$$E(u_n(t), B_{r_1} \setminus B_{r_0}) \geq 4\pi - \frac{\varepsilon_1}{2}\,.$$

By a similar reasoning, and since we have $h_n(1,t) = \pi/2$ for all t and n, we conclude that the map $h_n(t)$, restricted to $[r_1,1]$ covers the interval $[\pi/2, \pi - \sqrt{\frac{\varepsilon_1}{4\pi}}]$, and thus the map $u_n(t)$, restricted to the annulus $B \setminus B_{r_1}$ covers the lower hemisphere with a cap of radius $\sqrt{\frac{\varepsilon_1}{4\pi}}$ removed from the south pole. It follows that we have

$$E(u_n(t), B \setminus B_{r_1}) \geq 2\pi - \pi\left(\sqrt{\frac{\varepsilon_1}{4\pi}}\right)^2 \geq 2\pi - \frac{\varepsilon_1}{4}\,,$$

leaving the energy

$$\begin{aligned} E(u_n(t), B_{r_0}) &= E(u_n(t)) - \big(E(u_n(t), B \setminus B_{r_1}) + E(u_n(t), B_{r_1} \setminus B_{r_0})\big) \\ &\leq 6\pi - (6\pi - \frac{3\varepsilon_1}{4}) = \frac{3\varepsilon_1}{4} < \varepsilon_1\,, \end{aligned}$$

for the ball B_{r_0} for any $t \in [t_0,T]$, any $n \geq N$, as claimed. □

Proof of Theorem 6.15, completed. In view of Lemma 6.16, from Lemma 6.10 we obtain uniform bounds for $\nabla^2 u_n$ in $L^2(B \times [t_0, T])$ for any $t_0 > 0$. Together with the smooth estimates on $B \times]0, T]$ away from $x = 0$ these estimates suffice to pass to the limit in Equation (6.14) for u_n and we obtain a solution u to this equation which splits off a bubble of area 4π when t decreases to 0. Moreover, $u(t) \to u_0$ weakly in $H^1(B)$ and hence strongly in $L^2(B)$ as $t \downarrow 0$. We may therefore shift time by 1 and join this solution of (6.14) with the constant solution $u(t) \equiv u_0$ for $0 \leq t \leq 1$ at time $t = 1$ to obtain a weak solution of (6.14) on the interval $[0, 1 + T]$ satisfying (6.19). In view of the inequality $E(u(t)) \leq 6\pi$, which by Lemma 6.8 must be strict for $t > 1$, again the flow u cannot become singular at any time $t > 1$ and therefore can be smoothly extended for all $t > 1$. This finishes the proof. □

6.17 Uniqueness issues. Theorem 6.15 also shows that if we drop the requirement that $E(u(t))$ be non-increasing, the initial value problem for (6.14) with data u_0 has more than just the constant solution $u(t) \equiv u_0$; in fact, since we may shift time by an arbitrary amount, Theorem 6.15 gives infinitely many distinct solutions that exhibit reverse bubbling at an arbitrary time $t_0 > 0$.

On the other hand, results of Rivière [1] and Freire [1], [2] establish uniqueness for the Cauchy problem (6.1), (6.2) also in the "energy class" of weak solutions u to (6.1) such that $E(u(t))$ is bounded and $\partial_t u \in L^2\big(\Sigma \times [0, T]\big)$, provided that the energy $E(u(t))$ is a non-increasing function of t. In particular, in view of the smoothing property of the flow (6.1) for short time, their results give a new proof of Hélein's [1] regularity result for weakly harmonic maps of surfaces. Topping [1] conjectured that uniqueness holds in the energy class if we impose suitable hypotheses to rule out reverse bubbling, for instance, if we suppose that

$$\limsup_{t \downarrow t_0} E(u(t)) < E(u(t_0)) + \varepsilon_0 \,, \tag{6.24}$$

where ε_0 is defined in Theorem 6.6. As shown by Rupflin [1], this conjecture holds true if, in addition to (6.24), we assume that the function $E(u(t))$ is of locally bounded variation, or if the constant ε_0 in (6.24) is replaced by a possibly smaller constant $\varepsilon_1 > 0$, depending only on the target manifold.

The preceding remarks refer to arbitrary smooth and closed target manifolds N. In the special case when $N = S^2$, from a physical interpretation of the Dirichlet energy $E(u, \Omega)$ as the Frank-Oseen energy of a nematic liquid crystal contained in a cylinder of unit length with cross-section Ω, Bertsch et al. [1] conjecture that there is a unique "physical time" $t_1 \geq t_0$ when the energy concentrated in a spherical bubble separating at time t_0 from a flow solution u of (6.14) may be "released" as the bubble is re-attached. It may be conjectured that suitable approximations, like the Ginzburg-Landau approximation studied by Harpes [1], can give rise to such uniqueness results for the flow (6.14) in a restricted energy class of solutions that allows both forward and backward bubbling.

6.18 Energy Quantization. At a singular time $\bar{t} < \infty$ the flow constructed in Theorem 6.6 splits off a finite number of bubbles $\bar{u}_j \colon S^2 \to N$, $j = 1, \ldots, k$, and we have the estimate

$$\lim_{t \uparrow \bar{t}} E(u(t)) \geq E(u(\bar{t})) + \sum_{j=1}^{k} E(\bar{u}_j)\,. \tag{6.25}$$

Similar to the quantization result for the Yamabe equation that we obtained in Theorem 3.1, we might expect (6.25) to hold with equality. When $N = S^n$, this was actually shown by Jie Qing [2]. His result was generalized to arbitrary target manifolds by C. Wang [1] and Ding-Tian [1], independently, with later improvements by Lin-Wang [1].

The analogous quantization result

$$\lim_{m \to \infty} E(u_m) = E(u_0) + \sum_{j=1}^{k} E(\bar{u}_j) \tag{6.26}$$

for sequences of harmonic maps $u_m \rightharpoonup u_0$ weakly in $H^1(B, N)$ with bubbles $\bar{u}_j \colon S^2 \to N$, $j = 1, \ldots, k$, was shown by Parker [1]. Parker's result was extended to Palais-Smale sequences (u_m) with

$$\tau(u_m) = \Delta u_m - A(u_m)(\nabla u_m, \nabla u_m) \to 0 \text{ in } L^2$$

and bounded energy by Qing-Tian [1]. By the energy inequality for the heat flow, Lemma 6.8, from (6.26) we then also obtain the corresponding energy quantization in the case when the heat flow concentrates as $t \to \infty$, for suitable sequences $t_m \to \infty$.

6.19 Remarks. (1°) Under suitable hypotheses Topping [3] shows exponential L^2-convergence as $t \to \infty$ of the heat flow (6.1) for harmonic maps $S^2 \to S^2$ in the presence of "bubbling at infinity", and exponential H^1-convergence locally away from any bubble points of the flow.
(2°) Results analogous to Theorem 6.6 hold for a number of planar geometric flows and may be applied to prove the existence of minimal surfaces or, more generally, of surfaces of constant mean curvature with free boundaries; see Struwe [16].
(3°) See Schoen-Uhlenbeck [1], [2], Struwe [14], Chen-Struwe [1], Coron [3], Coron-Ghidaglia [1], Chen-Ding [1] for results on harmonic maps and the evolution problem (6.1), (6.2) in the case $\dim(\Sigma) > 2$. A survey of results related to harmonic maps can be found in Eells-Lemaire [1], [2]. An overview of developments for the evolution problem is given in Struwe [20], [24].

Appendix A

Here, we collect without proof a few basic results about Sobolev spaces. A general reference to this topic is Gilbarg-Trudinger [1], or Adams [1].

Sobolev Spaces

Let Ω be a domain in $\mathbb{R}^n$. For $u \in L^1_{loc}(\Omega)$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, with $|\alpha| = \sum_{j=1}^n \alpha_j$, define the distibutional derivative $D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u$ by letting

$$< \varphi, D^\alpha u >= \int_\Omega (-1)^{|\alpha|} u \, D^\alpha \varphi \, dx, \tag{A.1}$$

for all $\varphi \in C_0^\infty(\Omega)$. We say $D^\alpha u \in L^p(\Omega)$, if there is a function $g_\alpha \in L^p(\Omega)$ satisfying

$$< \varphi, D^\alpha u >=< \varphi, g_\alpha >= \int_\Omega \varphi g_\alpha \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$. In this case we identify $D^\alpha u$ with $g_\alpha \in L^p(\Omega)$.

For $k \in \mathbb{N}_0$, $1 \le p \le \infty$, define the space

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega) \text{ for all } \alpha \colon |\alpha| \le k \right\},$$

with norm

$$\|u\|^p_{W^{k,p}} = \sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p}, \quad \text{if } 1 \le p < \infty,$$

respectively, with norm

$$\|u\|_{W^{k,\infty}} = \max_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty}.$$

Note that the distributional derivative (A.1) is continuous with respect to weak convergence in $L^1_{loc}(\Omega)$.

Many properties of $L^p(\Omega)$ carry over to $W^{k,p}(\Omega)$.

A.1 Theorem. *For any $k \in \mathbb{N}_0$, $1 \le p \le \infty$, $W^{k,p}(\Omega)$ is a Banach space. $W^{k,p}(\Omega)$ is reflexive if and only if $1 < p < \infty$. Moreover, $W^{k,2}(\Omega)$ is a Hilbert space with scalar product*

$$(u, v)_{W^{k,2}} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u \, D^\alpha v \, dx \ ,$$

inducing the norm above.

For $1 \le p < \infty$, $W^{k,p}(\Omega)$ also is separable. In fact, we have the following result due to Meyers and Serrin; see Adams [1; Theorem 3.16].

A.2 Theorem. *For any $k \in \mathbb{N}_0, 1 \le p < \infty$, the subspace $W^{k,p} \cap C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

The completion of $W^{k,p} \cap C^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $H^{k,p}(\Omega)$. By Theorem A.2, $W^{k,p}(\Omega) = H^{k,p}(\Omega)$. In particular, if $p = 2$ it is customary to use the latter notation.

Finally, $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$; in particular, $H_0^{k,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^{k,2}(\Omega)$, with dual $H^{-k}(\Omega)$. $\mathcal{D}^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{D^{k,p}}^p = \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p}^p \ .$$

Hölder Spaces

A function $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is Hölder continuous with exponent $\beta > 0$ if

$$[u]^{(\beta)} = \sup_{x \ne y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\beta} < \infty.$$

For $m \in \mathbb{N}_0$, $0 < \beta \le 1$, denote

$$C^{m,\beta}(\Omega) = \big\{u \in C^m(\Omega);\ D^\alpha u \text{ is Hölder continuous with exponent } \beta \text{ for all } \alpha\colon |\alpha| = m\big\} \ .$$

If Ω is relatively compact, $C^{m,\beta}(\bar\Omega)$ becomes a Banach space in the norm

$$\|u\|_{C^{m,\beta}} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty} + \sum_{|\alpha|=m} [D^\alpha u]^{(\beta)}.$$

The space $C^{m,\beta}(\Omega)$ on an open domain $\Omega \subset \mathbb{R}^n$ carries a Fréchet space topology, induced by the $C^{m,\beta}$-norms on compact sets exhausting Ω. Finally, we may set $C^{m,0}(\Omega) := C^m(\Omega)$. Observe that for $0 < \beta \le 1$ smooth functions are *not* dense in $C^{m,\beta}(\bar\Omega)$.

Embedding Theorems

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. X is (continuously) embedded into Y (denoted $X \hookrightarrow Y$) if there exists an injective linear map $i\colon X \to Y$ and a constant C such that

$$\|i(x)\|_Y \le C\|x\|_X, \quad \text{for all } x \in X.$$

In this case we will often simply identify X with the subspace $i(X) \subset Y$.

X is compactly embedded into Y if i maps bounded subsets of X into relatively compact subsets of Y.

For the spaces that we are primarily interested in we have the following results. First, from Hölder's inequality we obtain:

A.3 Theorem. *For $\Omega \subset \mathbb{R}^n$ with Lebesgue measure $\mathcal{L}^n(\Omega) < \infty$, $1 \le p < q \le \infty$, we have $L^q(\Omega) \hookrightarrow L^p(\Omega)$. This ceases to be true if $\mathcal{L}^n(\Omega) = \infty$.*

For Hölder spaces, by the theorem of Arzéla-Ascoli we have the following compactness result; see Adams [1; Theorems 1.30, 1.31].

A.4 Theorem. *Suppose Ω is a relatively compact domain in $\mathbb{R}^n$, and let $m \in \mathbb{N}_0$, $0 \le \alpha < \beta \le 1$. Then $C^{m,\beta}(\bar{\Omega}) \hookrightarrow C^{m,\alpha}(\bar{\Omega})$ compactly.*

Finally, for Sobolev spaces we have (see Adams [1; Theorem 5.4]):

A.5 Theorem *(Sobolev embedding theorem).* *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $k \in \mathbb{N}$, $1 \le p \le \infty$. Then the following holds:*
(1°) If $k\,p < n$, we have $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q \le \frac{n\,p}{n-k\,p}$; the embedding is compact, if $q < \frac{n\,p}{n-k\,p}$.
(2°) If $0 \le m < k - \frac{n}{p} < m+1$, $m \in \mathbb{N}_0$, we have $W^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\bar{\Omega})$ for $0 \le \alpha \le k - m - \frac{n}{p}$; the embedding is compact, if $\alpha < k - m - \frac{n}{p}$.

Compactness of the embedding $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q < \frac{n\,p}{n-k\,p}$ is a consequence of the Rellich-Kondrakov theorem; see Adams [1; Theorem 6.2].
Theorem A.5 is valid for $W_0^{k,p}(\Omega)$-spaces on arbitrary bounded domains Ω.

Density Theorem

By Theorem A.2, Sobolev functions can be approximated by functions enjoying any degree of smoothness in the interior of Ω. Some regularity condition on the boundary $\partial\Omega$ is necessary if smoothness up to the boundary is required:

A.6 Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1, and let $k \in \mathbb{N}$, $1 \le p < \infty$. Then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.*

More generally, it suffices that Ω has the segment property; see for instance Adams [1; Theorem 3.18].

Trace and Extension Theorems

For a domain $\Omega \subset\subset \mathbb{R}^n$ with C^k-boundary $\partial\Omega = \Gamma$, $k \in \mathbb{N}$, $1 < p < \infty$, denote as $W^{k-\frac{1}{p},p}(\Gamma)$ the space of "traces" $u|_\Gamma$ of functions $u \in W^{k,p}(\Omega)$.

If $k = 1$ we think of $W^{1-\frac{1}{p},p}(\Gamma)$ as the set of equivalence classes $\big\{\{u\} + W_0^{1,p}(\Omega); u \in W^{1,p}(\Omega)\big\}$, endowed with the trace norm

$$\|u|_\Gamma\|_{W^{1-\frac{1}{p},p}(\Gamma)} = \inf\big\{\|v\|_{W^{1,p}(\Omega)}; u - v \in W_0^{1,p}(\Omega)\big\}.$$

By this definition, $W^{1-\frac{1}{p},p}(\Gamma)$ is a Banach space.

In particular, if $k = 1$ and $p = 2$, the trace operator $u \mapsto u|_{\partial\Omega}$ is a linear isometry of the (closed) orthogonal complement of $H_0^{1,2}(\Omega)$ in $H^{1,2}(\Omega)$ onto $H^{\frac{1}{2},2}(\Gamma)$. By the open mapping theorem this provides a bounded "extension operator" $H^{\frac{1}{2},2}(\Gamma) \to H^{1,2}(\Omega)$. For general $p > 1$ (and $k = 1$) we have:

A.7 Theorem. *For any $\Omega \subset\subset \mathbb{R}^n$ with C^1-boundary Γ, $1 < p < \infty$ there exists a continuous linear extension operator* $\mathrm{ext}: W^{1-\frac{1}{p},p}(\Gamma) \to W^{1,p}(\Omega)$ *such that* $\big(\mathrm{ext}(u)\big)\big|_\Gamma = u$, *for all* $u \in W^{1-\frac{1}{p},p}(\Gamma)$.

See Adams [1; Theorem 7.53 and 7.55].

Covering $\partial\Omega = \Gamma$ by coordinate patches and defining the Lebesgue space $L^p(\Gamma)$ as before via such charts (see Adams [1; 7.51]), an equivalent norm for $W^{s,p}(\Gamma)$, where $0 < s < 1$, is given by

$$\|u\|_{\tilde{W}^{s,p}} = \left\{ \|u\|_{L^p}^p + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^p}{|x-y|^{n-1+sp}}\, dx\, dy \right\}^{1/p};$$

see Adams [1; Theorem 7.48].

From this, the following may be deduced:

A.8 Theorem. *Suppose $\Omega \subset\subset \mathbb{R}^n$ is a domain with C^1-boundary Γ, $1 < p < \infty$. Then $W^{1,p}(\Gamma) \hookrightarrow W^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow L^p(\Gamma)$ and both embeddings are compact.*

In particular, we have

$$H^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega) \tag{A.2}$$

compactly, for any bounded domain of class C^1.

Poincaré Inequality

For a bounded domain Ω of diameter d and $u \in H_0^{1,2}(\Omega)$ there holds

$$\int_\Omega |u|^2 dx \le d^2 \int_\Omega |\nabla u|^2\, dx\ . \tag{A.3}$$

It suffices to consider $\Omega \subset [0,d]\times\mathbb{R}^{n-1} = S$, $u \in C_0^\infty(\Omega) \subset C_0^\infty(S)$. Then (A.3) follows immediately from Hölder's inequality and the mean value theorem.

More generally, we state:

A.9 Theorem. *For any bounded domain Ω of class C^1 there exists a constant $c = c(\Omega)$ such that for any $u \in H^{1,2}(\Omega)$ we have*

$$\int_\Omega |u|^2\, dx \le c \int_\Omega |\nabla u|^2\, dx + c \int_{\partial\Omega} |u|^2\, do\ .$$

Proof. The argument is modeled on Nečas [1; p. 18 f.]. Suppose by contradiction that for a sequence (u_m) in $H^{1,2}(\Omega)$ there holds

$$\|u_m\|_{L^2(\Omega)}^2 \ge m \left(\|\nabla u_m\|_{L^2(\Omega)}^2 + \|u_m\|_{L^2(\partial\Omega)}^2 \right)\ . \tag{A.4}$$

By homogeneity, we may normalize $\|u_m\|^2_{L^2(\Omega)} = 1$. But then (u_m) is bounded in $H^{1,2}(\Omega)$, and we may assume that $u_m \to u$ weakly. Moreover, by Theorem A.5.(1°), it follows that $u_m \to u$ strongly in $L^2(\Omega)$ and by (A.2) also $u_m\,|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^2(\partial\Omega)$.

But (A.4) also implies that $\nabla u_m \to 0$ in $L^2(\Omega)$, and $u_m\,|_{\partial\Omega} \to 0$ in $L^2(\partial\Omega)$. Hence, $u \in H^{1,2}_0(\Omega)$; moreover, $\nabla u = 0$. By (A.3) therefore, $u \equiv 0$. But $\|u\|_{L^2(\Omega)} = \lim_{m\to\infty} \|u_m\|_{L^2(\Omega)} = 1$. Contradiction. □

In the same spirit the following variant of Poincaré's inequality may be derived.

A.10 Theorem. *Let $A_R = B_{2R}(0) \setminus B_R(0) \subset \mathbb{R}^n$ denote the annulus of size R in $\mathbb{R}^n$. There exists a constant $c = c(n,p)$ such that for any $R > 0$, any $u \in H^{1,p}(A_R)$ there holds*

$$\int_{A_R} |u - \overline{u}_R|^p \, dx \le c\, R^p \int_{A_R} |\nabla u|^p \, dx \ ,$$

where $\overline{u}_R$ denotes the mean of u over the annulus A_R.

Proof. Scaling with R, we may assume that $R = 1$, $A_R = A_1 =: A$. Moreover, it suffices to consider $\overline{u} = \overline{u}_1 = 0$. If for a sequence (u_m) in $H^{1,p}(A)$ with $\overline{u}_m = 0$ we have

$$1 = \int_A |u_m|^p \, dx \ge m \int_A |\nabla u_m|^p \, dx \ ,$$

by Theorem A.5 we conclude that $u_m \to u \equiv const. = \overline{u} = 0$ in $L^p(A)$. Contradiction. □

Appendix B

In this appendix we recall some fundamental estimates for elliptic equations. A basic reference is Gilbarg-Trudinger [1].

On a domain $\Omega \subset \mathbb{R}^n$ we consider second-order elliptic differential operators of the form

$$L\,u = -a_{ij}\frac{\partial^2}{\partial x_j \partial x_j}u + b_i\frac{\partial}{\partial x_i}u + cu, \tag{B.1}$$

or in divergence form

$$L\,u = -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}u\right) + c\,u\ , \tag{B.2}$$

with bounded coefficients $a_{ij} = a_{ji}, b_i$, and c satisfying the ellipticity condition

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

with a uniform constant $\lambda > 0$, for all $\xi \in \mathbb{R}^n$. By convention, repeated indices are summed from 1 to n. The standard example is the Laplace operator $L = -\Delta$. If $a_{ij} \in C^1$, then any operator of type (B.2) also falls into category (B.1) with $b_j = -\frac{\partial}{\partial x_i}a_{ij}$.

Schauder Estimates

Let us first consider the (classical) C^α-setting; see Gilbarg-Trudinger [1; Theorems 6.2, 6.6].

B.1 Theorem. *Let L be an elliptic operator of type (B.1), with coefficients of class C^α, and let $u \in C^2(\Omega)$. Suppose $L\,u = f \in C^\alpha(\bar\Omega)$. Then $u \in C^{2,\alpha}(\Omega)$, and for any $\Omega' \subset\subset \Omega$ we have*

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\bar\Omega)}\right). \tag{B.3}$$

If in addition Ω is of class $C^{2+\alpha}$, and if $u \in C^o(\bar\Omega)$ coincides with a function $u_o \in C^{2+\alpha}(\bar\Omega)$ on $\partial\Omega$, then $u \in C^{2,\alpha}(\bar\Omega)$ and

$$\|u\|_{C^{2,\alpha}(\bar\Omega)} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\bar\Omega)} + \|u_o\|_{C^{2,\alpha}}(\bar\Omega)\right) \tag{B.4}$$

with constants C possibly depending on L, Ω, n, α, and – in case of (B.3) – on Ω'.

L^p-theory

For solutions in Sobolev spaces the Calderón-Zygmund inequality is the counterpart of the Schauder estimates for classical solutions; see Gilbarg-Trudinger [1; Theorems 9.11, 9.13].

B.2 Theorem. *Let L be elliptic of type (B.1) with continuous coefficients a_{ij}. Suppose $u \in W^{2,p}_{loc}(\Omega)$ satisfies $L\,u = f$ in Ω with $f \in L^p(\Omega)$, $1 < p < \infty$. Then for any $\Omega' \subset\subset \Omega$ we have*

$$\|u\|_{W^{2,p}(\Omega')} \le C\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right) . \tag{B.5}$$

If in addition Ω is of Class $C^{1,1}$, and if there exists a function $u_o \in W^{2,p}(\Omega)$ such that $u - u_o \in H^{1,p}_o(\Omega)$, then

$$\|u\|_{W^{2,p}(\Omega)} \le C\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|u_o\|_{W^{2,p}(\Omega)}\right) . \tag{B.6}$$

The constants C may depend on L, Ω, n, p, and – in case of (B.5) – on Ω'.

Weak Solutions

Let L be elliptic of divergence type (B.2), $f \in H^{-1}(\Omega)$. A function $u \in H^{1,2}_0(\Omega)$ weakly solves the equation $L\,u = f$ if

$$\int_\Omega \left(a_{ij}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x_j}\varphi + c\,u\varphi\right)dx - \int_\Omega f\varphi dx = 0, \text{ for all } \varphi \in C^\infty_o(\Omega).$$

The integral

$$\mathcal{L}(u,\varphi) = \int_\Omega \left(a_{ij}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x_j}\varphi + c\,u\varphi\right)\,dx$$

continuously extends to a symmetric bilinear form $\mathcal{L}$ on $H^{1,2}_0(\Omega)$, the Dirichlet form associated with the operator L.

A Regularity Result

As an application we consider the equation

$$-\Delta u = g(\cdot, u) \text{ in } \Omega, \tag{B.7}$$

on a domain $\Omega \subset \mathbb{R}^n$, with a Caratheodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$; that is, assuming $g(x,u)$ is measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}$. Moreover, we will assume that g satisfies the growth condition

$$|g(x,u)| \le C\big(1 + |u|^p\big) , \tag{B.8}$$

where $p \le \frac{n+2}{n-2}$, if $n \ge 3$. By (B.8) and Theorem A.5, for any $u \in H^{1,2}(\Omega)$ the composed function $g\big(\cdot, u(\cdot)\big) \in H^{-1}(\Omega)$; see also Theorem C.2. The following estimate is essentially due to Brezis-Kato [1], based on Moser's [1] iteration technique.

B.3 Lemma. *Let Ω be a domain in $\mathbb{R}^n$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \Omega$ there holds*

$$(1^\circ) \qquad |g(x,u)| \le a(x)\big(1+|u|\big)$$

with a function $0 \le a \in L^{n/2}_{loc}(\Omega)$. Also let $u \in H^{1,2}_{loc}(\Omega)$ be a weak solution of equation (B.7). Then $u \in L^q_{loc}(\Omega)$ for any $q < \infty$. If $u \in H^{1,2}_0(\Omega)$, and $a \in L^{n/2}(\Omega)$, then $u \in L^q(\Omega)$ for any $q < \infty$.

Proof. Choose $\eta \in C^\infty_0(\Omega)$ and for $s \ge 0$, $L \ge 0$ let $\varphi = \varphi_{s,L} = u \min\big\{|u|^{2s}, L^2\big\}\eta^2 \in H^{1,2}_0(\Omega)$, with supp $\varphi \subset\subset \Omega$. Testing (B.7) with φ, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u|^2 \min\big\{|u|^{2s}, L^2\big\}\eta^2 \, dx + \frac{s}{2}\int_{\{x\in\Omega;\, |u(x)|^s \le L\}} \big|\nabla\big(|u|^2\big)\big|^2 |u|^{2s-2}\eta^2 \, dx \\
&\quad \le -2\int_\Omega \nabla u \, u \, \min\big\{|u|^{2s}, L^2\big\}\nabla\eta\eta \, dx \\
&\qquad + \int_\Omega a\big(1+2|u|^2\big)\min\big\{|u|^{2s}, L^2\big\}\eta^2 \, dx \\
&\quad \le \frac{1}{2}\int_\Omega |\nabla u|^2 \min\big\{|u|^{2s}, L^2\big\}\eta^2 \, dx + c\int_\Omega |u|^2 \min\big\{|u|^{2s}, L^2\big\}|\nabla\eta|^2 \, dx \\
&\qquad + 3\int_\Omega a|u|^2 \min\big\{|u|^{2s}, L^2\big\}\eta^2 \, dx + \int_\Omega |a|\eta^2 \, dx \, .
\end{aligned}$$

Suppose $u \in L^{2s+2}(\mathrm{supp}(\eta))$. Then for any $K \ge 1$ with constants c depending on the L^{2s+2}-norm of u, restricted to $\mathrm{supp}(\eta)$, there holds

$$\begin{aligned}
\int_\Omega \big|\nabla\big(u\min\big\{|u|^s, L\big\}\eta\big)\big|^2 \, dx &\le c + c\cdot\int_\Omega a|u|^2\min\big\{|u|^{2s}, L^2\big\}\eta^2 \, dx \\
&\le c + cK\int_\Omega |u|^2\min\big\{|u|^{2s}, L^2\big\}\eta^2 \, dx \\
&\qquad + c\int_{\{x\in\Omega;\, a(x)\ge K\}} a|u|^2\min\big\{|u|^{2s}, L^2\big\}\eta^2 \, dx \\
&\le c(1+K) + \left(c\cdot\int_{\{x\in\Omega;\, a(x)\ge K\}} a^{n/2} \, dx\right)^{2/n} \\
&\qquad \times \left(\int_\Omega \big|u\min\big\{|u|^s, L\big\}\eta\big|^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}} \\
&\le c(1+K) + c_1\varepsilon(K)\cdot\int_\Omega \big|\nabla\big(u\min\big\{|u|^s, L\big\}\eta\big)\big|^2 \, dx \, ,
\end{aligned}$$

where

$$\varepsilon(K) = \left(\int_{\{x\in\Omega;\, a(x)\ge K\}} a^{n/2} \, dx\right)^{2/n} \to 0 \qquad (K\to\infty) \, .$$

Fix K such that $c_1\varepsilon(K) = \frac{1}{2}$ and observe that for this choice of K (and s as above) we now may conclude that

$$\int_\Omega \left|\nabla\big(u\min\{|u|^s, L\}\eta\big)\right|^2 dx \le c(1+K)$$

remains uniformly bounded in L. Hence we may let $L \to \infty$ to derive that

$$|u|^{s+1}\eta \in H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)\,.$$

That is, whenever $u \in L^{2s+2}_{loc}(\Omega)$ we find that $u \in L^{\frac{(2s+2)n}{n-2}}_{loc}(\Omega)$. Now iterate, letting $s_0 = 0$, $s_i + 1 = (s_{i-1}+1)\frac{n}{n-2}$, if $i \ge 1$, to obtain the conclusion of the lemma. If $u \in H_0^{1,2}(\Omega)$, we may let $\eta = 1$ to obtain that $u \in L^q(\Omega)$ for all $q < \infty$. □

To apply Lemma B.3, note that, if $u \in H^{1,2}_{loc}(\Omega)$ weakly solves (B.7) with a Carathéodory function g with polynomial growth

$$|g(x,u)| \le C\big(1 + |u|^p\big)\ ,$$

and if $p \le \frac{n+2}{n-2}$ for $n > 2$, then assumption (1°) of Lemma B.3 is satisfied with

$$a(x) = \frac{|g\big(x, u(x)\big)|}{1+|u(x)|} \in L^{n/2}_{loc}(\Omega)\,.$$

By Lemma B.3, therefore, $u \in L^q_{loc}(\Omega)$, for any $q < \infty$. In view of our growth condition for g this implies that $-\Delta u = g(u) \in L^q_{loc}(\Omega)$ for any $q < \infty$. Thus, by the Caldéron-Zygmund inequality, Theorem B.2, $u \in W^{2,q}_{loc}(\Omega)$, for any $q < \infty$, whence also $u \in C^{1,\alpha}_{loc}(\Omega)$ by the Sobolev embedding theorem, Theorem A.5, for any $\alpha < 1$. Moreover, if $u \in H_0^{1,2}(\Omega)$, and if $\partial\Omega \in C^2$, by the same token it follows that $u \in W^{2,q} \cap H_0^{1,2}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$. Now we may proceed using Schauder theory. In particular, if g is Hölder continuous, then $u \in C^2(\Omega)$ and is a non-constant, classical C^2-solution of Equation (B.7). Finally, if g and $\partial\Omega$ are smooth, higher regularity (up to the boundary) can be obtained by iterating the Schauder estimates.

Maximum Principle

A basic tool for proving existence of solutions to elliptic boundary value problems in Hölder spaces is the maximum principle.

We state this in a form due to Walter [1; Theorem 2], allowing for more general coefficients c in the operator L than in classical versions.

B.4 Theorem. *Suppose L is elliptic of type (B.1) on a domain Ω and suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$L\,u \geq 0 \text{ in } \Omega, \text{ and } u \geq 0 \text{ on } \partial\Omega.$$

Moreover, suppose there exists $h \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that

$$L\,h \geq 0 \text{ in } \Omega, \text{ and } h > 0 \text{ on } \Omega.$$

Then either $u > 0$ in Ω, or $u = \beta h$ for some $\beta \leq 0$.

In particular, let L be given by (B.2) with coefficients $a_{ij} \in C^{1,\alpha}(\bar{\Omega}), c \in C^{\alpha}(\bar{\Omega})$. Then L is self-adjoint and possesses a complete set of eigenfunctions (φ_j) in $H_0^{1,2}(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ with eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. Moreover, φ_1 has constant sign, say, $\varphi_1 > 0$ in Ω. Suppose that the first Dirichlet eigenvalue

$$\lambda_1 = \inf_{u \neq 0} \frac{(L\,u, u)_{L^2}}{(u, u)_{L^2}} \geq 0.$$

Then in Theorem B.4 we may choose $h = \varphi_1$, and the theorem implies that any solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of $Lu \geq 0$ in Ω, and such that $u \geq 0$ on Ω either is positive throughout Ω or vanishes identically.

The strong maximum principle is based on the Hopf boundary maximum principle; see Walter [1; p. 294]:

B.5 Theorem. *Let L be elliptic of type (B.1) on the ball $B = B_R(0) \subset \mathbb{R}^n$, with $c \geq 0$. Suppose $u \in C^2(B) \cap C^0(\overline{B})$ satisfies $Lu \geq 0$ in B, $u \geq 0$ on ∂B, and $u \geq \gamma > 0$ in $B_\rho(0)$ for some $\rho < R, \gamma > 0$. Then there exists $\delta = \delta(L, \gamma, \rho, R) > 0$ such that*

$$u(x) \geq \delta\big(R - |x|\big) \qquad \text{in } B\ .$$

In particular, if $u \in C^2(B) \cap C^1(\overline{B})$ and if $u(x_0) = 0$ for some $x_0 \in \partial B$, then the interior normal derivative of u at the point x_0 is strictly positive.

Weak Maximum Principle

For weak solutions of elliptic equations we have the following analogue of Theorem B.4.

B.6 Theorem. *Suppose L is elliptic of type (B.2) and suppose the Dirichlet form of $\mathcal{L}$ is positive definite on $H_0^{1,2}(\Omega)$ in the sense that*

$$\mathcal{L}(u, u) > 0 \text{ for all } u \in H_0^{1,2}(\Omega),\ u \neq 0.$$

Then, if $u \in H^{1,2}(\Omega)$ weakly satisfies $L\,u \geq 0$ in the sense that

$$\mathcal{L}(u, \varphi) \geq 0 \quad \text{for all non-negative } \varphi \in H_0^{1,2}(\Omega),$$

and $u \geq 0$ on $\partial\Omega$, it follows that $u \geq 0$ in Ω.

Proof. Choose $\varphi = u_- = \max\{-u, 0\} \in H_0^{1,2}(\Omega)$. Then

$$0 \leq \mathcal{L}(u, u_-) = -\mathcal{L}(u_-, u_-) \leq 0$$

with equality if and only if $u_- \equiv 0$; that is $u \geq 0$. □

Theorem B.6 can be used to strengthen the boundary maximum principle Theorem B.5:

B.7 Theorem. *Let L satisfy the hypotheses of Theorem B.6 in $\Omega = B_R(0) = B \subset \mathbb{R}^n$ with coefficients $a_{ij} \in C^1$. Suppose $u \in C^2(B) \cap C^1(\overline{B})$ satisfies $Lu \geq 0$ in B, $u \geq 0$ on ∂B and $u \geq \gamma > 0$ in $B_\rho(0)$ for some $\rho < R, \gamma > 0$. Then there exists $\delta = \delta(L, \gamma, \rho, R) > 0$ such that $u(x) \geq \delta\big(R - |x|\big)$ in B.*

Proof. We adapt the proof of Walter [1; p. 294]. For large $C > 0$ the function $v = \exp\big(C\big(R^2 - |x|^2\big)\big) - 1$ satisfies $Lv \leq 0$ in $B \setminus B_\rho(0)$. Moreover, for small $\varepsilon > 0$ the function $w = \varepsilon v$ satisfies $w \leq u$ for $|x| \leq \rho$ and $|x| = R$. Hence, Theorem B.6 – applied to $u - w$ on $B \setminus B_\rho(0)$ – shows that $u \geq w$ in $B \setminus B_\rho(0)$. □

Application

As an application, consider the operator $L = -\Delta - \delta$, where $\delta < \lambda_1$, the first Dirichlet eigenvalue of $-\Delta$ on Ω. Let $u \in H_0^{1,2}(\Omega)$ or $u \in C^2(\Omega) \cap C^o(\bar{\Omega})$ weakly satisfy $L\,u \leq C_o$ in $\Omega, u \leq 0$ on $\partial\Omega$, and choose $v(x) = C\big(C - |x - x_0|^2\big)$ with $x_0 \in \Omega$ and C sufficiently large to achieve that $v > 0$ on $\bar{\Omega}$ and $L\,v \geq C_0$. Then $w = v - u$ satisfies

$$L\,w \geq 0 \text{ in } \Omega, \quad w > 0 \text{ on } \partial\Omega\ ,$$

and hence w is non-negative throughout Ω. Thus

$$u \leq v \qquad \text{in } \Omega\ .$$

More generally, results like Theorem B.4 or B.5 can be used to obtain L^∞- or even Lipschitz a-priori bounds of solution to elliptic boundary value problems by comparing with suitably constructed "barriers".

Appendix C

In this appendix we discuss the issue of (partial) differentiability of variational integrals of the type

$$E(u) = \int_\Omega F\big(x, u(x), \nabla u(x)\big)\, dx, \tag{C.1}$$

where $u \in H^{1.2}(\Omega)$, for simplicity. Differentiability properties will crucially depend on growth conditions for F.

Fréchet-differentiability

A functional E on a Banach space X is Fréchet-differentiable at a point $u \in X$ if there exists a bounded linear map $DE(u) \in X^*$, called the differential of E at u, such that

$$\frac{|E(u+v) - E(u) - DE(u)v|}{\|v\|_X} \to 0$$

as $\|v\|_X \to 0$. E is of class C^1, if the map $u \mapsto DE(u)$ is continuous.

C.1 Theorem. *Let $\Omega \subset\subset \mathbb{R}^n$. Suppose $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in $x \in \Omega$, continuously differentiable in $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$, with $F_u = \frac{\partial}{\partial u}F$, $F_p = \frac{\partial}{\partial p}F$, and the following growth conditions are satisfied:*
(1°) $|F(x,u,p)| \le C(1 + |u|^{s_1} + |p|^2)$, where $s_1 \le \frac{2n}{n-2}$, if $n \ge 3$,
(2°) $|F_u(x,u,p)| \le C(1 + |u|^{s_2} + |p|^{t_2})$, where $t_2 < 2$, if $n \le 2$, respectively, where $s_2 \le \frac{n+2}{n-2}, t_2 \le \frac{n+2}{n}$, if $n \ge 3$,
(3°) $|F_p(x,u,p)| \le C(1 + |u|^{s_3} + |p|)$, where $s_3 \le \frac{n}{n-2}$, if $n \ge 3$.
Then (C.1) defines a C^1-functional E on $H^{1,2}(\Omega)$. Moreover, $DE(u)$ is given by

$$< v, DE(u) >= \int_\Omega \big(F_u(x,u,\nabla u)v + F_p(x,u,\nabla u)\cdot\nabla v\big)\, dx\ .$$

Theorem C.1 applies for example to the functional

$$G(u) = \int_\Omega |u|^p\, dx$$

with $p \le \frac{2n}{n-2}$, if $n \ge 3$, or to Dirichlet's integral

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx.$$

Theorem C.1 rests on a result by Krasnoselskii [1; Theorem I.2.1]. For simplicity, we state this result for functions $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$. To ensure measurability of composed functions $g\big(x, u(x)\big)$, with $u \in L^p$, we assume $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function; that is, g is measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}^m$.

C.2 Theorem. *Suppose $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition*
(1°) $|g(x,u)| \le C(1 + |u|^s)$ for some $s \ge 1$.
Then the operator

$$u \mapsto \big(g(\cdot, u(\cdot)\big)$$

is continuous from $L^{sp}(\Omega)$ into $L^p(\Omega)$ for any $p, 1 \le p < \infty$.

Theorem C.2 asserts that Nemitskii operators – that is, evaluation operators like (C.1) – are continuous if they are bounded. For nonlinear operators this is quite remarkable.

Using this result, Theorem C.1 follows quite naturally from the Sobolev embedding theorem, Theorem A.5. To get a flavor of the proof, we establish continuity of the derivative of a functional E as in Theorem C.1. For $u_0, u \in H_0^{1,2}(\Omega)$, we estimate

$$\begin{aligned}
\|DE(u) - DE(u_0)\| &= \sup_{\substack{v \in H_0^{1,2} \\ \|v\|_{H_0^{1,2}} \le 1}} |< v, DE(u) - DE(u_0) >| \\
&\le \sup_v \int_\Omega |F_u(x,u,\nabla u) - F_u(x,u_0,\nabla u_0)|\, |v|\, dx \\
&\quad + \sup_v \int_\Omega |F_p(x,u,\nabla u) - F_p(x,u_0,\nabla u_0)|\, |\nabla v|\, dx \\
&\le \sup_v \left(\int_\Omega |F_u(x,u,\nabla u) - F_u(x,u_0,\nabla u_0)|^{\frac{2n}{n+2}}\, dx \right)^{\frac{n+2}{2n}} \cdot \left(\int_\Omega |v|^{\frac{2n}{n-2}}\, dx \right)^{\frac{n-2}{2n}} \\
&\quad + \sup_v \left(\int_\Omega |F_p(x,u,\nabla u) - F_p(x,u_0,\nabla u_0)|^2\, dx \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |\nabla v|^2\, dx \right)^{\frac{1}{2}}
\end{aligned}$$

if $n \ge 3$ – which we will assume from now on for simplicity. Now, by Theorem A.5 the integrals involving v are uniformly bounded for $v \in H_0^{1,2}(\Omega)$ with $\|v\|_{H_0^{1,2}} \le 1$. By our growth conditions (2°) and (3°), moreover, F_u (respectively F_p) can be estimated like

$$|F_u(x,u,\nabla u)|^{\frac{2n}{n+2}} \le C \left(1 + |u|^{\frac{2n}{n-2}} + |\nabla u|^2\right),$$

and by Theorem C.2 it follows that $DE(u) \to DE(u_0)$, if $u \to u_0$, as desired.

Natural Growth Conditions

Conditions (1°)–(3°) of Theorem C.1 require a special structure of the function F; for instance, terms involving $|\nabla u|^2$ cannot involve coefficients depending on u. Consider for example the functional in Section I.1.5, given by

$$F(x,u,p) = g_{ij}(u)p_ip_j.$$

Note that F_u has the same growth as F with respect to p. More generally, for analytic functions F such that

$$|p|^2 \le F(x,u,p) \le C\big(|u|\big)\big(1+|p|^2\big) \tag{C.2}$$

one would expect the following growth conditions:

$$F_u(x,u,p) \le C\big(|u|\big)\big(1+|p|^2\big) \tag{C.3}$$

$$F_p(x,u,p) \le C\big(|u|\big)\big(1+|p|\big) \tag{C.4}$$

for $x \in \Omega$, $u \in \mathbb{R}$, and $p \in \mathbb{R}^n$.

Under these growth assumptions, in general a functional E given by (C.1) cannot be Fréchet-differentiable in $H^{1,2}(\Omega)$ any more. However, minimizers (in $H_0^{1,2}(\Omega)$, say) still may exist, compare Theorem I.1.5. Is it still possible to derive necessary conditions in the form of Euler-Lagrange equations? – The answer to this question may be positive, if we consider only a restricted set of minimizers and a narrower class of "testing functions", that is of admissible variations:

C.3 Theorem. *Suppose E is given by (C.1) with a Carathéodory function F, of class C^1 in u and p, satisfying the natural growth conditions (C.2)–(C.4). Then, if $u, \varphi \in H^{1,2} \cap L^\infty(\Omega)$, the directional derivative of E at u in direction φ exists and is given by:*

$$\frac{d}{d\varepsilon}E(u+\varepsilon\varphi)\big|_{\varepsilon=0} = \int_\Omega \big(F_u(x,u,\nabla u)\varphi + F_p(x,u,\nabla u)\cdot\nabla\varphi\big)\,dx\ .$$

In particular, at a minimizer $u \in H^{1,2} \cap L^\infty$ of E, with F satisfying (C.2)–(C.4), the Euler-Lagrange equations are weakly satisfied in the sense that

$$\int_\Omega \big(F_u(x,u,\nabla u)\cdot\varphi + F_p(x,u,\nabla u)\nabla\varphi\big)\,dx = 0$$

holds for all $\varphi \in H_0^{1,2} \cap L^\infty(\Omega)$.

Note that the assumption $u \in L^\infty$ often arises naturally, as in Theorem I.1.5. Sometimes, boundedness of minimizers may also be derived a posteriori. For further details, we refer to Giaquinta [1] or Morrey [4]. The question of differentiability of functionals in general is quite subtle, as is illustrated by an example of Ball and Mizel [1].

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Index

Approximate solutions 33, 69
Area 6, 19, 220
  Co-area formula 43
Barrier 273
  (see also: Sub- and super-solution)
Bifurcation 180 ff.
Caldéron-Zygmund inequality 268 f.
Carathéodory function 275
Category 100
  (see also: Index theory)
Change in topology 169
Characteristic function 6
Coercive
  functional 4
  operator 60
Compactness
  bounded compactness 2
  concentration-compactness 37 f., 39 f., 44 f.
  compensated compactness 25 ff.
  local compactness 41, 175, 226
  global compactness in the critical case 184, 231
    (see also: Palais-Smale condition)
  of a sequence of measures 39
Concentration-function (of measure) 40
Conformal
  conformal group of the disk 20 f.
  conformal invariance 20, 194, 232
  conformal Laplace operator 197
  conformality relation 20
  ε-conformality theorem of Morrey 20
Convexity 9, 12 f., 21, 25, 54, 57, 61 f., 103, 105, 162 ff.
  polyconvex 25
  quasiconvex 54
Critical point (value) 1
  at infinity 169
  in convex sets 164
  of mountain pass type 143
  of non-differentiable functional 152
  saddle point 1, 50, 54, 73, 74, 76, 77, 87, 108, 151
Deformation lemma 81 ff.
  for C^1-functionals on Banach spaces 83
  for C^1-functionals on Finsler manifolds 87
  for non-differentiable functionals 153 f., 158
  on convex sets 165
Developing map 197 f.
Dichotomy (of a sequence of measures) 40
Dirichlet integral 20
Dual variational problem 62, 67
Eigenvalue for Dirichlet problem
  Courant-Fischer characterization 97
  Rayleigh-Ritz characterization 14f.
  Weyl asymptotic formula 118
Elliptic equations 14 ff., 16 ff., 32 ff., 98 ff., 110 ff., 116 ff., 120 ff., 128 f., 147 ff., 150 ff., 166 ff., 170 ff.
  degenerate elliptic equations 4 ff., 183
  on unbounded domains 36 ff., 150 ff.
  with critical growth 170 ff.
Energy
  energy functional 25, 71 f., 150, 231
  energy inequality 199, 204, 238, 243
  stored energy 25
  energy surface 61
Epigraph 57 f.
Equivariant 82, 84, 86, 94
Euler-Lagrange equations 1
Finsler manifolds 85 f.
Fréchet-differential 274
Functional at infinity 36
Genus 94 f.
Geodesics 61, 89
  closed geodesics on spheres 89 ff.
Gradient 84
  gradient-flow 84, 135
  (see also: Pseudo-gradient)
Group action 82, 84, 86, 94
Hamiltonian systems 60 ff., 103 ff., 124, 130 ff., 137 ff., 150
Hardy space 35, 224
Harmonic map 8, 71 ff., 169, 231 ff.
  evolution problem 238 ff.
Harmonic sphere 240
Index theory 94 ff., 99
  Benci-index 101 ff.
  Krasnoselskii genus 94 ff.
  Ljusternik-Schnirelman category 100 f.
  pseudo-index 101
Intersection lemma 113
Invariant
  under flow 87
  under group action (see: Equivariant)
Isoperimetric inequality 43, 224
Legendre condition 13
Legendre-Fenchel transform 58 f., 63 f.
Limiting problem 170, 184

Linking 125 ff.
examples of linking sets 125 ff., 134
Lower semi-continuity 2 ff., 8 ff., 25, 51, 58, 70
Maximum principle 271 ff.
Mean curvature equation 169, 220 ff.
Measure
compactness of a sequence of measures 39
concentration function of measure 40
dichotomy of a sequence of measures 40
vanishing of a sequence of measures 39
Minimal surface 6 f., 19 ff., 169
parametric minimal surface 19 f.
minimal cones 7
minimal partitioning surfaces 6 f.
Minimax principle 74, 87 ff., 96 f.
Courant-Fischer 97
Palais 87
Minimizer 1, 51, 70, 144, 166
Minimizing sequence 3, 53, 55 ff., 70
Monotone operator 60
Monotonicity (of index) 99
Mountain pass lemma 74, 76, 108 ff., 112
Palais-Smale condition 77 ff.
Condition (C) 78
Condition (P.-S.) 78
Cerami's variant 80
for non-differentiable functionals 152
local 177
on convex sets 164
Palais-Smale sequence 54, 78
Penalty method 69 f.
Periodic solutions
of Hamiltonian systems 61 ff., 103 ff., 130 ff., 137 ff., 150
of semilinear wave equation 65 ff., 124, 150
with prescribed minimal period 104 f.
Perimeter (of a set) 6
Perron's method 16 ff.
Plateau problem 19 ff., 214
boundary condition 19
Pohožaev identity 155 f., 171
Poincaré inequality 266 f.
Pseudo-gradient flow 84
(see also: Deformation lemma)
Pseudo-gradient vector field 81, 86
for non-differentiable functionals 153
on convex sets 164
Pseudo-Laplace operator (p-Laplacian) 5, 183
Regular point (value) 1, 164
Regularity theory 16, 31, 57, 268 ff.
for minimal surfaces 24
for the constant mean curvature equation 223
in elasticity 31
partial regularity for evolution of harmonic maps 245 ff., 248 f., 253 ff.
Rellich-Kondrakov theorem 265
Schauder estimates 268
Schwarz-symmetrization 42
Separation of spheres 169, 239 ff.
Sobolev embedding (inequality) 42 ff., 170 ff., 242, 265 ff.
density of smooth maps in Sobolev spaces 235 f.
Sub-additivity (of index) 99
Sub-differential 58
Sub-solution 16 f.
Super-solution 17
Supervariance (of index) 99
Support
support hyperplane 58
support function 58
Symmetry 16, 36, 169
Symmetry group (see: Group action)
Symplectic structure 60
Technique
Fatou-lemma technique 34
hole-filling technique 56
Vanishing (of a sequence of measures) 39
Variational inequality 13, 166
Volume 220, 223 ff.
Wave equation 65 ff., 124, 150
Yamabe problem 18, 194 ff.